Multiplicative Sumudu Transform and its Applications

Aarif Hussain Bhat, Javid Majid, Imtiyaz Ahmad Wani
Research Scholar, Research Scholar, Research Scholar,
School of Mathematics and Allied Sciences
Jiwaji University, Gwalior, M.P India

ABSTRACT

This work is aimed to give some basic definitions of multiplicative Sumudu transform and its properties. The multiplicative Sumudu transform is obtained by using Sumudu transform and its properties in a classical analysis as basis. Solving some multiplicative differential equations by this transform is used as an application.

Keywords

Sumudu transform, Multiplicative derivative, Multiplicative Sumudu transform, Multiplicative integral.

1. Introduction

The key to understand multiplicative calculus is a formal substitution, whereby one replaces addition and subtraction by multiplication and division, respectively. Michael Grossman and Robert Katz began their development of Non- Newtonian calculus on 14 july 1967.There are infinitely many multiplicative non-newtonian calculi, including the geometric and bio-geometric calculus. It has been applied in a variety of scientific, engineering and mathematical fields **[1].** However **D.Stanley** called the geometric analysis as multiplicative calculus **[2].**Further study to multiplicative calculus was given by **D.Campell [3]**, then **Bashirov et al. [4]** has given concepts of non-Newtonian calculus and its applications including properties of derivative and integral operators of non-Newtonian calculus. With the passage of time some researchers **[5-12]** have proved that multiplicative calculus is very helpful in solving problems related to science and engineering fields.

In the present paper, we will study multiplicative Samudu transform and its applications. In order to study this we give definition of multiplicative Samudu transform, Secondly we will give some basic properties of multiplicative samudu transform as corresponding to classical samudu. In the end we find solution of some multiplicative differential equations by applying multiplicative samudu transform.

2 .Multiplicative Derivative.

In this section we will present some basic definition and properties of multiplicative derivative which can be seen in **[2-5]**

Definition2.1: Let $g : R \to R^+$ be a positive function. The Multiplicative derivative of the function g is given by

$$\frac{d^*g}{dt}(t) = g^*(t) = \lim_{h\to 0}\left(\frac{g(t+h)}{g(t)}\right)^{\frac{1}{h}} \qquad (2.1)$$

Let us assume that the function g is a positive then using properties of the classical derivative we can write multiplicative derivative as

$$\frac{d^*g}{dt}(t) = g^*(t) = e^{\frac{g'(t)}{g(t)}} = e^{(\ln og)'(t)} \qquad (2.2)$$

for $(\ln og)(t) = \ln(g(t))$.

Definition 2.2: If g is a positive function and if g^* is a multiplicative derivative of g, now if the function g^* has also multiplicative derivative which is given by g^{**} and it is called second order multiplicative derivative of g. Similarly we can define $g^{*(n)}$. which is called nth order multiplicative derivative of g. By repeating multiplicative differentiation operation n-times, we will find nth order multiplicative derivative of the positive function g at the point t which defined as

$$g^{*(n)} = e^{(\ln o g)^{n}(t)} \tag{2.3}$$

Theorem 2.1: Let g and h be differentiable with the multiplicative derivative .If c be arbitrary constant, then $c.g, gh, g+h, {}^{g}/_{h}, g^{h}$ functions are differentiable with the multiplicative derivative and their multiplicative derivatives can be shown as

1.$(c.g)^*(t) = g^*(t)$,

2.$(g.h)^*(t) = g^*(t).h^*(t)$,

3.$(g+h)^*(t) = g^*(t)^{\frac{g(t)}{g(t)+h(t)}}\, h^*(t)^{\frac{h(t)}{g(t)+h(t)}}$,

4.$\left({}^{g}/_{h}\right)^* = g^*(t)/h^*(t)$,

5.$(g^h)^*(t) = g^*(t)^{h(t)} g(t)^{h'(t)}$

Theorem 2.2: If a positive function g is differentiable with the multiplicative derivative at the point t, then it is differentiable in the classical sense and the relation between these two derivatives can be shown as

$$g'(t) = g(t) \ln g^*(t) \tag{2.4}$$

Theorem 2.3: $g^*(t) = 1$ for $\forall\ t \in (a,b) \Leftrightarrow g(t) = c > 0$ is fixed function in open interval (a,b)

Theorem 2.4: Let h be differentiable in meaning of the multiplicative derivative, g be differentiable in the classical sense. If

$$g(t) = (hok)(t),$$

Then, it can be written by

$$g^*(t) = [h^*(k(t))]^{k'(t)} \tag{2.5}$$

Theorem 2.5: Let g be a positive function then, $g^*(t) = 1 \Leftrightarrow g'(t) = 0$

Multiplicative integrals

Definition2.3: A multiplicative integral is also defined in **[4]** for positive bounded functions and if g is Riemann integrable on [a,b], then

$$\int_a^b g(t)^{dt} = exp\left(\int_a^b \ln g(t)\, dt\right) = e^{\int_a^b (\ln g(t))dt} \tag{2.6}$$

This multiplicative integral has the following properties

(a) $\int_a^b ((g(t)^k)^{dt} = \left(\int_a^b g(t)^{dt}\right)^k$, Where $k \in R$

$$(b)\ \int_a^b (g(t)h(t))^{dt} = \int_a^b g(t)^{dt} \int_a^b h(t)^{dt}$$

$$(c)\ \int_a^b \left(\frac{g(t)}{h(t)}\right)^{dt} = \frac{\int_a^b (g(t))^{dt}}{\int_a^b (h(t))^{dt}}$$

$$(d)\ \int_a^b g(t)^{dt} = \int_a^c g(t)^{dt} \int_c^b g(t)^{dt} \qquad a \le c \le b$$

Where g *and* h are multiplicative integrable [a, b].

3. The Multiplicative Sumudu Transform

In this section we will give the features of this new transform by defining multiplicative Sumudu transform with the help of Sumudu transform of classical analysis.

Definition 3.1 Let $f(t)$ be a positive definite function given on the closed interval $[0,\infty)$. Then multiplicative samudu transform of $f(t)$ is defined as

$$\mathcal{S}_m\{f(t)\} = F_m(u) = \frac{1}{u}\int_0^\infty f(t)^{e^{\left(\frac{-t}{u}\right)}dt} = e^{\frac{1}{u}\int_0^\infty e^{\frac{-t}{u}}\ln f(t)dt} = e^{S\{\ln f(t)\}} \tag{3.1}$$

Here, multiplicative integral, which is defined as

$$\int_0^\infty f(t)^{dt} = e^{\int_0^\infty \ln f(t)dt} \tag{3.2}$$

has been used.

According to definition of multiplicative sumudu transform, multiplicative samudu transform of some basic functions can be given as the following expression.

$$\mathcal{S}_m\{1\} = F_m(u) = e^{\frac{1}{u}\int_0^\infty \ln(1)e^{\frac{-t}{u}}dt} = 1 \tag{3.3}$$

$$\mathcal{S}_m\{e^t\} = e^{S\{\ln e^t\}} = e^{S\{t\}} = e^u \tag{3.4}$$

$$\mathcal{S}_m\{e^{at}\} = e^{S\{\ln e^{at}\}} = e^{\frac{1}{1-au}} \tag{3.5}$$

$$\mathcal{S}_m\{e^{cosat}\} = e^{S\{\ln e^{cosat}\}} = e^{\frac{1}{1+a^2u^2}} \tag{3.6}$$

$$\mathcal{S}_m\left\{e^{\frac{1}{a}sinat}\right\} = e^{S\{\ln e^{\frac{1}{a}sinat}\}} = e^{\frac{u}{1+a^2u^2}} \tag{3.7}$$

$$\mathcal{S}_m\{e^{coshat}\} = e^{S\{\ln e^{coshat}\}} = e^{\frac{1}{1-a^2u^2}} \tag{3.8}$$

$$\mathcal{S}_m\left\{e^{\frac{1}{a}sinhat}\right\} = e^{S\{\ln e^{\frac{1}{a}sinhat}\}} = e^{\frac{u}{1-a^2u^2}} \tag{3.9}$$

Theorem 3.1 (Multiplicative Linearity Property)

Multiplicative Sumudu transform is multiplicatively linear, in other words, if c_1, c_2 are arbitrary exponents and f_1 , f_2 are two given functions , which have multiplicative sumudu transform then

$$\mathcal{S}_m\{{f_1}^{c_1}{f_2}^{c_2}\} = \mathcal{S}_m\{f_1\}^{c_1}\,\mathcal{S}_m\{f_2\}^{c_2} \tag{3.10}$$

Theorem 3.2 (Multiplicative first shifting property)

Let multiplicative Sumudu transform of function $f(t)$ be $\mathcal{S}_m\{f(t)\} = F_m(u)$, then

$$\mathcal{S}_m\{f(t)^{e^{at}}\} = e^{\frac{1}{1-ua}} F_m\left\{\frac{u}{(1-ua)}\right\} \tag{3.11}$$

Proof: $\mathcal{S}_m\{f(t)^{e^{at}}\} = e^{\frac{1}{u}\int_0^\infty \{\ln f(t)\}^{e^{at}} e^{\frac{-t}{u}} dt}$

$$= e^{\frac{1}{u}\int_0^\infty \ln f(t) e^{-t\left(\frac{1}{u}-a\right)} dt}$$

Now making substitution $-t\left(\frac{1}{u} - a\right) = x$

$$\mathcal{S}_m\{f(t)^{e^{at}}\} = e^{\frac{1}{1-ua}\int_0^\infty \ln f\left(\frac{xu}{1-ua}\right) e^{-x} dx}$$

$$\mathcal{S}_m\{f(t)^{e^{at}}\} = e^{\frac{1}{1-ua}} F_m\left\{\frac{u}{(1-ua)}\right\}$$

Theorem 3.3 (Multiplicative second first shifting property)

Let $\mathcal{S}_m\{G(t)\} = F_m(u)$ and $G(t) = \begin{cases} 1, & 0 < t < a \\ f(t-a), & t > a \end{cases}$ (3.12)

then $\mathcal{S}_m\{G(t)\} = F_m(u)^{e^{\frac{-a}{u}}}$ (3.13)

Proof: $\mathcal{S}_m\{G(t)\} = e^{\frac{1}{u}\int_0^\infty \ln G(t) e^{\frac{-t}{u}} dt}$

$$= e^{\frac{1}{u}\int_0^a \ln(1) e^{\frac{-t}{u}} dt + \frac{1}{u}\int_a^\infty \ln f(t-a) e^{\frac{-t}{u}} dt}$$

Here first integral is zero use substitution for 2nd integral we get

$$\mathcal{S}_m\{G(t)\} = e^{\frac{1}{u}\int_0^\infty \ln f(x) e^{\frac{-(a+x)}{u}} dx}$$

$$\mathcal{S}_m\{G(t)\} = e^{e^{\left(\frac{-a}{u}\right)}\frac{1}{u}\int_0^\infty \ln f(x) e^{\frac{-x}{u}} dx}$$

$$\mathcal{S}_m\{G(t)\} = e^{\left\{\frac{1}{u}\int_0^\infty \ln f(x) e^{\frac{-x}{u}} dx\right\}^{e^{\left(\frac{-a}{u}\right)}}}$$

$$\mathcal{S}_m\{G(t)\} = F_m(u)^{e^{\frac{-a}{u}}}$$

Theorem 3.4: (Multiplicative change of scale property)

Let $\mathcal{S}_m\{f(t)\} = F_m(u)$ then we have

$$\mathcal{S}_m\{f(at)\} = F_m(ua) \tag{3.14}$$

Proof: we know by definition

$$\mathcal{S}_m\{f(at)\} = e^{\frac{1}{u}\int_0^\infty \ln f(at) e^{\frac{-t}{u}} dt}$$

Now by making substitution $at = x$

$$\mathcal{S}_m\{f(at)\} = e^{\frac{1}{ua}\int_0^\infty \ln f(x) e^{\left(\frac{-x}{ua}\right)} dx}$$

$$\mathcal{S}_m\{f(at)\} = F_m(ua)$$

Theorem 3.5: Let $\mathcal{S}_m\{f(t)\} = F_m(u)$ then

$$\mathcal{S}_m\left\{f(t)^{\left(\frac{t-u}{u^2}\right)^n}\right\} = F_m{}^{*n}(u) \tag{3.15}$$

Proof: we will prove this theorem by induction

$$\mathcal{S}_m\{f(t)\} = F_m(u) = \frac{1}{u}\int_0^\infty f(t)^{e^{\left(\frac{-t}{u}\right)}dt}$$

$$\mathcal{S}_m\{f(t)\} = F_m(u) = e^{\frac{1}{u}\int_0^\infty \ln f(t) e^{\frac{-t}{u}} dt}$$

Consequently taking multiplicative derivative of this expression we have

$$F_m{}^*(u) = \frac{d^*}{du^*}\left\{e^{\frac{1}{u}\int_0^\infty \ln f(t) e^{\frac{-t}{u}} dt}\right\} = e^{\frac{d}{du}\left\{e^{\frac{1}{u}\int_0^\infty \ln f(t) e^{\frac{-t}{u}} dt}\right\}}$$

$$= e^{\left(\frac{t-u}{u^2}\right)\frac{1}{u}\int_0^\infty \ln f(t) e^{\frac{-t}{u}} dt} = e^{\frac{1}{u}\int_0^\infty \{\ln f(t)\}^{\left(\frac{t-u}{u^2}\right)} e^{\frac{-t}{u}} dt}$$

$$F_m{}^*(u) = \mathcal{S}_m\left\{[f(t)]^{\left(\frac{t-u}{u^2}\right)}\right\} = \mathcal{S}_m\left\{[f(t)]^{\left(\frac{t-u}{u^2}\right)^1}\right\}$$

and we get the following equality

$$\mathcal{S}_m\left\{[f(t)]^{\left(\frac{t-u}{u^2}\right)^1}\right\} = F_m{}^{*1}(u)$$

Let us assume the hypothesis holds for the case $n = k$. then we have

$$F_m{}^{*k}(u) = e^{\frac{1}{u}\int_0^\infty \left(\frac{t-u}{u^2}\right)^k \ln f(t)\, e^{\frac{-t}{u}} dt} = \mathcal{S}_m\left\{[f(t)]^{\left(\frac{t-u}{u^2}\right)^k}\right\}$$

Now if the multiplicative derivative is taken again for the last equation, we have

$$F_m{}^{*k+1}(u) = \mathcal{S}_m\left\{f(t)^{\left(\frac{t-u}{u^2}\right)^{k+1}}\right\} = e^{\frac{1}{u}\int_0^\infty \left(\frac{t-u}{u^2}\right)^{k+1} \ln f(t)\, e^{\frac{-t}{u}} dt}$$

Similarly if we repeat multiplicative derivative n- times we will get

$$F_m{}^{*n}(u) = \mathcal{S}_m\left\{[f(t)]^{\left(\frac{t-u}{u^2}\right)^n}\right\}$$

This proves the theorem.

Theorem 3.6 (Multiplicative convolution theorem).

If $\mathcal{S}_m{}^{-1}\{F(u)\} = f(t)$ and $\mathcal{S}_m{}^{-1}\{G(u)\} = g(t)$, then

$$\mathcal{S}_m{}^{-1}\{F_m(u)^{G(u)}\} = \frac{1}{u}\int_0^t f(x)^{g(t-x)^{dx}} \tag{3.16}$$

Proof: Applying multiplicative sumudu transform to integral $\frac{1}{u}\int_0^t f(x)^{g(t-x)^{dx}}$ we get,

$$\mathcal{S}_m\left\{\frac{1}{u}\int_0^t f(x)^{g(t-x)^{dx}}\right\} = \mathcal{S}_m\left\{e^{\frac{1}{u}\int_0^t g(t-x)\ln f(x)dx}\right\}$$

$$= e^{S\left\{\ln\left\{e^{\frac{1}{u}\int_0^t g(t-x)\ln f(x)dx}\right\}\right\}} = e^{S\left\{\frac{1}{u}\int_0^t g(t-x)\ln f(x)dx\right\}}$$

From the convolution property of classical analysis, we have

$$\mathcal{S}_m\left\{\frac{1}{u}\int_0^t f(x)^{g(t-x)^{dx}}\right\} = e^{S\{\ln f(x)\}\, S\{g(x)\}}$$

$$\mathcal{S}_m\left\{\frac{1}{u}\int_0^t f(x)^{g(t-x)^{dx}}\right\} = e^{S\{\ln f(x)\}^{S\{g(x)\}}}$$

$$\mathcal{S}_m\left\{\frac{1}{u}\int_0^t f(x)^{g(t-x)^{dx}}\right\} = F_m(u)^{G(u)}$$

$$\frac{1}{u}\int_0^t f(x)^{g(t-x)^{dx}} = \mathcal{S}_m{}^{-1}\left\{F_m(u)^{G(u)}\right\}$$

Definition 3.2 Let $f(t)$ be a positive definite function given on the closed interval [0,∞) and $\ln f(t)$ be a function given on the interval [0,∞) if their exist positive constant $t_{0,}k$ and α such that

$$|f(t)| \le ke^{e^{\alpha t}} \tag{3.17}$$

For $t > t_0$ then f is said to be of α-double exponential order.

Theorem 3.7(Existence of multiplicative sumudu transform)

Let $f(t)$ be a positive definite function which is of α-double exponential order for $t > t_0$ given on the closed interval [0, ∞) then for s > α, $\mathcal{S}_m\{f(t)\}$ exists.

Proof: we will show that for s > α, integral $\frac{1}{u}\int_0^\infty \ln f(t)e^{\frac{-t}{u}}dt$ is convergent, to do this we will divide integral into two integrals as below

$$\frac{1}{u}\int_0^\infty \ln f dt = \frac{1}{u}\int_0^{t_0} \ln f(t)e^{\frac{-t}{u}}dt + \frac{1}{u}\int_{t_0}^\infty \ln f(t)e^{\frac{-t}{u}}dt \tag{3.18}$$

The first integral in above equation is convergent because of $\ln f(t)$ and thus $\ln f(t)e^{\frac{-t}{u}}$ is piece-wise continuous in the interval [0, t_0] .On the other hand, as $f(t)$ is of α-double exponential order

$$|f(t)| \le ke^{e^{\alpha t}}$$

For $t > t_0$, hence for all $t > t_0$

$$\left|\frac{1}{u}\ln f(t)e^{\frac{-t}{u}}\right| = \frac{1}{u}e^{\frac{-t}{u}}|\ln f(t)| \le \frac{1}{u}e^{\frac{-t}{u}}(\ln k + e^{\alpha t})$$

$$\frac{1}{u}\int_0^{\infty} \ln f(t) e^{\frac{-t}{u}} dt \leq \frac{1}{u}\int_{t_0}^{\infty} e^{\frac{-t}{u}} |\ln(k)| dt + \frac{1}{u}\int_{t_0}^{\infty} e^{\frac{-t}{u}+\alpha t} dt$$

$$= \ln k e^{\frac{t_0}{u}} + \frac{e^{-t_0\left(\frac{1-u\alpha}{u}\right)}}{(1-u\alpha)} < \infty, \text{for } s > \alpha$$

So the second integral of (3.18) is convergent we see that both the integrals on the RHS of exits. Thus for s$>\alpha$ $\mathcal{S}_m\{f(t)\}$ exists.

Theorem 3.8 Let $f(t)$ be a positive definite function which is of α-double exponential order for $t > t_0$, given the closed interval $[0,\infty)$ and $\ln f(t)$ be a piecewise continuous function defined on the interval $[0,\infty)$ then the following holds

$$\lim_{u\to\infty}\{f(t)\} = e^k \tag{3.19}$$

Proof: from the proof of theorem (3.7) we can write

$$\mathcal{S}_m\{f(t)\} \leq e^{\frac{1}{u}\int_0^{\infty} \ln k e^{e\alpha t} e^{\frac{-t}{u}} dt}$$

$$\leq e^{\frac{1}{u}\int_0^{\infty} \ln k e^{\frac{-t}{u}}\, dt + \frac{1}{u}\int_0^{\infty} e^{\left(\frac{-t}{u}+\alpha t\right)} dt}$$

$$\mathcal{S}_m\{f(t)\} \leq e^{\ln k\left(e^{\frac{-t_0}{u}}\right)}$$

For all s$>\alpha$, then taking limit from both sides as $u \to \infty$ we get

$$\lim_{u\to\infty}\{f(t)\} \leq e^{\lim_{u\to\infty} k\left(e^{\frac{-t_0}{u}}\right)} = e^k$$

Theorem 3.9 Let f be a function of α-double exponential order defined on the interval [0,A] and let f^* be a piece-wise continuous function defined on the interval [0,A], then samudu transform of multiplicative derivative is

$$\mathcal{S}_m\{f^*(t)\} = \frac{1}{f(0)} F(u)^{\frac{1}{u}} \tag{3.20}$$

For s $> \alpha$

Proof: $\mathcal{S}_m\{f^*(t)\} = \mathcal{S}_m\left\{e^{\frac{f'(t)}{f(t)}}\right\} = e^{\frac{1}{u}\int_0^{\infty}\frac{f'(t)}{f(t)} e^{\frac{-t}{u}} dt}$

$$= e^{\lim_{A\to\infty}\frac{1}{u}\int_0^{A}\frac{f'(t)}{f(t)} e^{\frac{-t}{u}} dt}$$

In the interval [0,A] denote the points of dis-continuities of function f^* by $t_0, t_1, t_2 \ldots\ldots\ldots t_n$ using the points as end-points of domain of integration ,we write the integral as

$$\frac{1}{u}\int_0^A \frac{f'(t)}{f(t)} e^{\frac{-t}{u}} dt = \frac{1}{u}\int_0^{t_1} \frac{f'(t)}{f(t)} e^{\frac{-t}{u}} dt + \frac{1}{u}\int_{t_1}^{t_2} \frac{f'(t)}{f(t)} e^{\frac{-t}{u}} dt + \cdots + \frac{1}{u}\int_{t_n}^{A} \frac{f'(t)}{f(t)} e^{\frac{-t}{u}} dt$$

Using integration by parts method separately to each term on the right hand side of this expression, we get

$$\frac{1}{u}\int_0^A \frac{f'(t)}{f(t)} e^{\frac{-t}{u}} dt = \frac{1}{u}\left\{ \left| e^{\frac{-t}{u}} \ln f(t) \right|_0^{t_1} + \left| e^{\frac{-t}{u}} \ln f(t) \right|_{t_1}^{t_2} + \cdots + \left| e^{\frac{-t}{u}} \ln f(t) \right|_{t_n}^{A} \right\}$$
$$+ \frac{1}{u^2}\left\{ \int_0^{t_1} e^{\frac{-t}{u}} \ln f(t) dt + \int_{t_1}^{t_2} e^{\frac{-t}{u}} \ln f(t) dt + \cdots + \int_{t_n}^{A} e^{\frac{-t}{u}} \ln f(t) dt \right\}$$

As $f(t)$ is continuous, domains of integration of the above expression can be combined in one domain .Thus we write

$$e^{\frac{1}{u}\int_0^A \frac{f'(t)}{f(t)} e^{\frac{-t}{u}} dt} = e^{\frac{1}{u}\left\{ e^{\frac{-t}{u}} \ln f(t) \Big|_0^A \right\} + \frac{1}{u^2}\left\{ \int_0^A e^{\frac{-t}{u}} \ln f(t) dt \right\}}$$

$$= e^{\frac{1}{u}\left\{ e^{\frac{-A}{u}} \ln f(A) - \ln f(0) \right\} + \frac{1}{u^2}\left\{ \int_0^A e^{\frac{-t}{u}} \ln f(t) dt \right\}}$$

as $A \to \infty, e^{\frac{-A}{u}} \ln f(A) \to 0$ and $e^{\frac{1}{u}e^{\frac{-A}{u}}} \to 1$

For $> \alpha$, then for $u > \alpha$,

we obtain $\mathcal{S}_m\{f^*(t)\} = e^{-\ln f(0)} e^{\left\{\frac{1}{u}\int_0^A f(t) e^{\frac{-t}{u}}\right\}^{\frac{1}{u}}}$

$$\mathcal{S}_m\{f^*(t)\} = \frac{1}{f(0)} F_m(u)^{\frac{1}{u}}$$

This proves the theorem, if we replace f by f^* in above theorem then multiplicative samudu transform of f^{**} is equal to

$$\mathcal{S}_m\{f^{**}(t)\} = \frac{F(u)^{\left(\frac{1}{u}\right)^2}}{f(0)^{\frac{1}{u}} f(0)} \tag{3.21}$$

Using induction we can obtain multiplicative sumudu transform of $f^{*(n)}$ as in the following result.

Result: Let $f, f^* \ldots f^{*(n-1)}$ be continuous function, f^{*n} be a piece-wise continuous function on the interval $0 \leq t \leq A$ also suppose that there exist positive real numbers k, α such that

$$|f(t)| \leq ke^{e^{\alpha t}}, |f^*(t)| \leq ke^{e^{\alpha t}} \ldots \left|f^{*(n-1)}(t)\right| \leq ke^{e^{\alpha t}}$$

For $t > t_0$ then for s$> \alpha$ multiplicative samudu transform of $f^{*n}(t)$ exists and can be calculated by the formula

$$\mathcal{S}_m\{f^{*n}(t)\} = \frac{F(u)^{\left(\frac{1}{u}\right)^n}}{f(0)^{\left(\frac{1}{u}\right)^{n-1}} f^*(0)^{\left(\frac{1}{u}\right)^{n-2}} \ldots\ldots\ldots\ldots f^{*n-1}(0)} \tag{3.22}$$

Theorem 3.10 Let f_1 and f_2 be positive definite continuous functions,

$$f_1 = f_2 \text{ if and only if } \mathcal{S}_m\{f_1\} = \mathcal{S}_m\{f_2\}.$$

Proof: $f_1 = f_2 \Leftrightarrow \ln f_1 = \ln f_2$ from the sumudu transform of classical analysis we get

$$\mathcal{S}\{\ln f_1\} = \mathcal{S}\{\ln f_2\} \quad \Leftrightarrow e^{\mathcal{S}\{\ln f_1\}} = e^{\mathcal{S}\{\ln f_2\}}, \text{ so } \mathcal{S}_m\{f_1\} = \mathcal{S}_m\{f_2\}.$$

Definition 3.3.if $F_m(u)$ is the multiplicative samudu transform of a continuous f, i.e,

$$\mathcal{S}_m\{f\} = F \tag{3.23}$$

${\mathcal{S}_m}^{-1}\{F\}$ is called as the inverse multiplicative samudu transform of F.

Theorem 3.11 Inverse multiplicative samudu transform is multiplicatively linear. In other words, if c_1, c_2 are arbitrary exponents and f_1 , f_2 are two given continuous functions which have multiplicative samudu transform F_1, F_2 respectively, then

$${\mathcal{S}_m}^{-1}\{{F_1}^{c_1}{F_2}^{c_2}\} = {\mathcal{S}_m}^{-1}\{F_1\}^{c_1}\ {\mathcal{S}_m}^{-1}\{F_2\}^{c_2} \tag{3.24}$$

Proof: Suppose f_1 and f_2 are continous functions such that

$$F_1 = \mathcal{S}_m\{f_1\} \quad \text{and} \quad F_2 = \mathcal{S}_m\{f_2\}$$

From the definition, we know that

$${\mathcal{S}_m}^{-1}\{F_1\} = f_1 \quad \text{and} \quad {\mathcal{S}_m}^{-1}\{F_2\} = f_2$$

Using the multiplicative linearity property of multiplicative samudu transform we have

$$\mathcal{S}_m\{{f_1}^{c_1}\ {f_2}^{c_2}\} = \mathcal{S}_m\{f_1\}^{c_1}\mathcal{S}_m\{f_2\}^{c_2} = {F_1}^{c_1}{F_2}^{c_2}$$

From the definition of inverse multiplicative sumudu transform we obtain

$${\mathcal{S}_m}^{-1}\{{F_1}^{c_1}{F_2}^{c_2}\} = {f_1}^{c_1}{f_2}^{c_2} = {\mathcal{S}_m}^{-1}\{F_1\}^{c_1}\ {\mathcal{S}_m}^{-1}\{F_2\}^{c_2}$$

4. Applications to Multiplicative ordinary differential equation

We have the following formula for multiplicative samudu transforms of multiplicative derivatives

$$\mathcal{S}_m\{f^{*n}(t)\} = \frac{F(u)^{\left(\frac{1}{u}\right)^n}}{f(0)^{\left(\frac{1}{u}\right)^{n-1}}\quad f^*(0)^{\left(\frac{1}{u}\right)^{n-2}}\quad \ldots\ldots\ldots\ldots f^{*n-1}(0)} \tag{3.25}$$

This formula includes multiplicative samudu transform of $f, f^* f^{**} \ldots\ldots\ldots. f^{*(n-1)}$ functions, So it can be used to obtain solution of initial value problem, particularly of multiplicative type with constant exponentials .we get solution by applying multiplicative fourier transform to both sides of equations of these problems.

For example, consider the second order ordinary differential equation

$$(y^{**})(y^*)^{a_1}(y)^{a_2} = 1 \tag{3.26}$$

$$y(0) = b_0, y^*(0) = b_1$$

Here a_{1}, a_2 and b_{0}, b_1 are constants .Applying multiplicative samudu transform to both sides to both sides and using the multiplicative linearity property of multiplicative sumudu transform, we get

$$\mathcal{S}_m\{y^{**}\}\,\mathcal{S}_m\{y^*\}^{a_1}\,\mathcal{S}_m\{y\}^{a_2} = \mathcal{S}_m\{1\}$$

Now using previous result we obtain

$$\left\{\frac{y(u)^{\left(\frac{1}{u}\right)^2}}{y(0)^{\frac{1}{u}}y^*(0)}\right\}\left\{\frac{y(u)^{\frac{1}{u}}}{y(0)}\right\}^{a_1}\{y(u)\}^{a_2}=1$$

If the above equation is rearranged, we can write

$$\frac{y(u)^{\frac{1}{u^2}+\frac{a_1}{u}+a_2}}{\{b_0\}^{\left(\frac{1}{u}+a_1\right)}b_1}=1$$

$$y(u)^{\frac{1}{u^2}+\frac{a_1}{u}+a_2}=\left\{b_1[b_0]^{\left(\frac{1}{u}+a_1\right)}\right\}$$

$$y(u)=\left\{b_1[b_0]^{\left(\frac{1}{u}+a_1\right)}\right\}^{\frac{u^2}{1+ua_1+u^2a_2}}$$

Consequently taking inverse multiplicative samudu transform we obtain

$$y(t)=\mathcal{S}_m{}^{-1}y(u)$$

This method can be applied to any order of linear differential equation with constant exponentials.

Example 4.1 consider the following multiplicative differential equation

$$y^{**}y=1 \tag{4.1}$$

With initial condition $y(0)=e, y*(0)=1$

Applying multiplicative sumudu transform to both sides and using the multiplicative linearity property of multiplicative samudu transform, we get

$$\mathcal{S}_m\{y^{**}\}\,\mathcal{S}_m\{y\}=\mathcal{S}_m\{1\}$$

Now using previous result we obtain

$$\left\{\frac{y(u)^{\left(\frac{1}{u}\right)^2}}{y(0)^{\frac{1}{u}}y^*(0)}\right\}\{y(u)\}=1$$

$$y(u)^{\left(\frac{1}{u}\right)^2}y(u)=e^{1/u}=e^{\left(\frac{u}{1+u^2}\right)}$$

$$y(t)=\mathcal{S}_m{}^{-1}\{e\}^{\left(\frac{u}{1+u^2}\right)}=e^{\sin(t)} \tag{4.2}$$

5. Conclusion

In this paper, we have defined multiplicative Samudu transform for positive definite functions. It has been demonstrated that this transform has basic properties such as linearity, convolution, shifting properties. The existence of Multiplicative Samudu transform has also been proved. Finally, it has been shown that this transform is an alternative method to find solutions of some multiplicative differential equations.

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