A Sequence of a Class of Parabolic Systems Gradient Estimates

Yousri S. M. Yasin[1], Abdelrahman M. M. Mahmoud [2], Mohammed A. Ali [2], Nagla T. M. Elhassan [2]

[1]Department of Mathematics, Faculty of Education, University of Zalingei, Sudan

[2]Department of Mathematics, Deanship of Preparatory year, Northern Borders University, Saudi Arabia.

Abstract

In this paper we generalize a class of parabolic problems whose model is the nonhomogeneous as a sequence of degenerate/singular parabolic $(2+\epsilon)$-Laplacian system

$$(u_r)_t - div(|Du_r|^{\epsilon}Du_r) = \text{div}(|F_r|^{\epsilon}F_r),$$

such that

$$F_r \in L^{1+2\epsilon}_{\text{loc}} \implies Du_r \in L^{1+2\epsilon}_{\text{loc}}, \quad \forall r \in \mathbb{N} \quad \forall\, \epsilon \geq 0.$$

Also, we treat system with discontinuous coefficients of vanishing mean oscillation (VMO) type.

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1. Introduction

We present Calderón-Zygmund estimates for weak solutions to a sequence of a class of degenerate/singular parabolic systems and equations, a prominent model example of which is the nonhomogeneous, parabolic $(2+\epsilon)$-Laplacian system

$$(u_r)_t - div(|Du_r|^{\epsilon}Du_r) = div(|F_r|^{\epsilon}F_r), \qquad \epsilon > \frac{-4}{n+2}, \quad r = 1,2,\dots,s, \tag{1.1}$$

considered in the cylindrical domain $C := \Omega \times [0,T)$. Here $\Omega \subset \mathbb{R}^n$ is a bounded open set, and $u_r \in C^0\big((0,T); L^2(\Omega,\mathbb{R}^N)\big) \cap L^{2+\epsilon}\big(0,T; W^{1,2+\epsilon}(\Omega,\mathbb{R}^N)\big)$, $N \geq 1$, while $F_r \in L^{2+\epsilon}(C,\mathbb{R}^{nN})$ Such a system is degenerate when $\epsilon > 0$, and singular when $\epsilon < 0$; the lower bound on the exponent $2+\epsilon$ assumed in (1.1) is standard in the theory of the parabolic $(2+\epsilon)$-Laplacian operator, and unavoidable for the type of regularity we consider here (see [26]).

For a class of a system (1.1) we prove that

$$F_r \in L^{1+2\epsilon}_{\text{loc}}(C,\mathbb{R}^{nN}) \implies Du_r \in L^{1+2\epsilon}_{\text{loc}}(C,\mathbb{R}^{nN}) \tag{1.2}$$

for any $\epsilon \geq 0$. In the elliptic, stationary case

$$div(|Du_r|^{\epsilon}Du_r) = \text{div}(|F_r|^{\epsilon}F_r), \tag{1.3}$$

the result in (1.2) was essentially obtained in [14] in the scalar case ($N = 1$), and in [10] for systems ($N > 1$). The extension to anisotropic elliptic equations with possibly discontinuous, vanishing mean oscillation (VMO) coefficients has been adopted in [19], [20], while a class of general non-linear elliptic equations and systems in divergence form, under non-standard growth conditions, has been treated in [1].

The authors of [16] proved higher integrability of the spatial gradient for solutions of general non-linear parabolic systems with $(2+\epsilon)$-growth including (1.1), introducing a localization method to overcome the lack of homogeneity of parabolic systems with $(2+\epsilon)$-growth when $\epsilon \neq 0$. They come up with a sort of reverse-type Hölder inequality, see also [17], [18]. The new ingredient offered by these authors is a suitable

application of DiBenedetto's intrinsic geometry method for degenerate / singular parabolic systems (see [8]), in the setting of Gehring-type estimates. Subsequently, the author of [22] considered higher integrability of the gradient of solutions to (1.1), assuming $F_r \in L^{\infty}$, and therefore in $L^{1+2\epsilon}$ for every $\epsilon > 0$.

We will use results and partially some methods adapted from [5] and [1] to be able to prove the natural integrability result in (1.2).

We observe that system (1.1) is non-linear in the gradient, and therefore the use of singular integrals is ruled out, while, since it is degenerate/singular, and scales differently in space and time, no maximal function operator is naturally associated with the problem. We therefore again adopt an intrinsic geometry viewpoint, arguing directly on certain Calderón-Zygmund-type covering arguments, and completely avoiding the use of the maximal function operator or of other harmonic analysis principles as the good - λ-inequality one. A peculiar aspect of our work, which allows us to treat the general situation considered here, is that instead of using the $C^{1,\alpha}$-estimates for the homogeneous ($F_r \equiv 0$) $(2+\epsilon)$-Laplacian systems, as done in [10], [11], [14], and [22] for both the elliptic and the parabolic cases, we use only the $C^{0,1}$-estimates (see [8]), which immediately exhibit the right scaling properties when considered on intrinsic cylinders and perfectly fit in this context. This is a natural attempt, since we want to prove $L^{1+2\epsilon}$-estimates for Du_r, whose limit case is indeed given by the $C^{0,1}$-estimates; anyway, the proof is quite delicate. An approach to gradient estimates for equations in divergence form, making use of $C^{0,1}$-estimates, and working via maximal functions, has been introduced in the elliptic, homogeneous case in [5]; such an approach works for (homogeneous) parabolic equations only when $\epsilon = 0$ (see [24]), again for the reasons explained above. As already mentioned, it is worth pointing out that we cannot use here the so called good-λ-inequality principle; we instead replace it with a new, direct argument that we like to call the large-M-inequality principle (see (4.55)). We like to mention that, apart from the different scaling procedures adopted for the singular and degenerate cases, the proof offered here does not distinguish between the cases $\epsilon < 0$ and $\epsilon \geq 0$.

Our results cover a sequence of a class of degenerate/singular parabolic systems, of the type

$$(u_r)_t - div(a_r(z_r)|Du_r|^{\epsilon}Du_r) = \operatorname{div}(|F_r|^{\epsilon}F_r), \tag{1.4}$$

whose coefficients $a_r(z_r) \equiv a_r(x_r, t_r)$ may be discontinuous in a VMO/bounded mean oscillation

(BMO) fashion (see Section 2), for which we still prove (1.2); furthermore, extensions involving operators different from the $(2+\epsilon)$-Laplacian are outlined in Section 5. We also derive natural and neat local estimates for solutions in the form of certain non-homogeneous reverse-type Hölder inequalities (see (2.5)). Here, the non-homogeneity of the estimates precisely reflects that of the system (space/time) via the scaling deficit exponent d introduced in (2.6).

The problem of deriving Calderón-Zygmund-type estimates for elliptic and parabolic equations, eventually with discontinuous coefficients, is a classical one, and already has a long tradition. In the elliptic and scalar case, it has been usually faced via harmonic analysis tools such as non-linear commutators (see [6]), Riezs transform (see [15]), or the maximal function operator (see [19]; see also [12], [23]). Parabolic equations with coefficients of VMO/BMO type have been treated only in the linear case, and in particular, again when $\epsilon = 0$, making use of harmonic analysis tools such as non-linear commutators (see [13]) and, more recently, of the maximal function operator (see [4]); needless to say, such ingredients are not available in the case of the evolutionary $(2+\epsilon)$-Laplacian operator.

2. Notations and results

By cylinder $Q_{z_r}(\theta,\varrho) \subset \mathbb{R}^{n+1}$ centered at the point $z_r \equiv (x_r, t_r) \in \mathbb{R}^{n+1}$ with $\theta, \varrho > 0$, we will always mean a set of the type $Q_{z_r}(\theta,\varrho) = B_{x_r}(\varrho) \times (t_r - \theta, t_r + \theta)$, where, as usual, $B_{x_r}(\varrho) \coloneqq \{y_r \in \mathbb{R}^N : |x_r - y_r| < \varrho\}$; with abuse of terminology, such cylinders also called cubes. As a partial exception we will write B^1 to denote the unit ball centered at the origin of $\mathbb{R}^N$. When not essential, the center of a cylinder is not specified that is, $Q(\theta,\varrho) \equiv Q_z(\theta,\varrho)$. In the case of the standard parabolic cylinders, that is, when $\theta = \varrho = R^2$, we simply write $Q_R = Q(R^2, R)$. The parabolic boundary $\partial_{2+\epsilon}Q$ of a cylinder $Q_{\tilde{z}_r}(\theta,\varrho)$ is the union of the lower base $B_{\tilde{x}_r}(\varrho) \times \{\tilde{t}_r - \theta\}$ and the side surface $\{|x_r - \tilde{x}_r| = \varrho\} \times \{\tilde{t}_r - \theta, \tilde{t}_r + \theta\}$. Adopting a usual convention, c_n will denote a constant whose value may change in any two occurrences, and only the relevant dependences will be specified, as for example, $c_n(\gamma,\epsilon)$; particular constants will be denoted by $c_{n+1}, \tilde{c}_n$ and the like. For the Lebesgue measure of a measurable set A, we employ either of the notations $|A| = meas(A)$; then we define the mean value on a cylinder $Q \subset \mathbb{R}^{n+1}$ of an integrable function $v_r \in L^1(Q)$ by

$$(v_r)_Q \equiv \sum_r \fint_Q v_r \, dx_r \coloneqq \frac{1}{Q}\sum_r \int_Q v_r \, dx_r$$

When $Q \equiv Q_R$ we will also employ the notation $(v_r)_R \equiv (v_r)_{Q_R}$.

We define the class of coefficients $a_r(z_r) \equiv a_r(x_r, t_r)$ which we use when treating systems of the type (1.4). In order to preserve the basic parabolicity properties of the systems, and allowing a degeneration caused only by the presence of the factor $|Du_r|^\epsilon$ in (1.4), we shall always assume that the function $a_r : C \longrightarrow \mathbb{R}$ satisfies

$$1 + \epsilon \le a_r(z_r) \le 1 + 2\epsilon, \quad \forall\, z_r \in C. \tag{2.1}$$

Definition 2.1 We say that a function $a_r(z_r)$ satisfies the strong VMO condition if

$$\lim_{R\to 0} \omega(R) = 0, \tag{2.2}$$

where

$$\omega(R) \coloneqq \sum_r \sup_{Q \Subset C} \fint_Q |a_r(z_r) - (a_r)_Q| \, dz_r \tag{2.3}$$

and the supremum is taken among all cylinders of the type $Q_{z_r}(\theta,\varrho)$ with $\varrho \le R$ and $\theta \le R^2$. We say that the function $a_r(z_r)$ satisfies the strong BMO condition if

$$[a_r]_{BMO} \coloneqq \sup_{R>0} \omega(R) < \infty.$$

To adapt to the nonlinear parabolic structure we are allowed to pick more cylinders with respect to a usual elliptic-style VMO/BMO definition (see [25]), allowing for the size of the space radius ϱ of the cylinder Q to be unrelated to the time height θ. This class includes, for instance, all continuous coefficients $a_r(z_r)$, and it is large enough to include many possibly discontinuous functions. For instance, in (1.4) we may take $a_r(x_r, t_r) = b(x_r)c(t_r)$, where both $b(x_r)$ and $c(t_r)$ are usual VMO/BMO functions, in Ω and $[0, T)$, respectively, and satisfying (2.1). The strong VMO/BMO condition is, in our opinion, the natural one in order to treat situations as in (1.4). Indeed, when dealing with partial differential equations (PDEs), especially elliptic and parabolic ones, the notion VMO/BMO is usually given using a family of cubes or cylinders that are relevant both for the scaling properties and for the geometry of the equation. Since the works of DiBenedetto (see [8] and references therein), it is known that the natural class of cylinders $Q(\theta,\varrho)$ occurring in connection with (1.1) is the one having the ratio ϱ/θ not related to the coefficient

$a_r(z_r)$ but depending on the solution u_r itself, via quantities like, for instance, $\left|(Du_r)_Q\right|^{\epsilon}$, which are a priori arbitrary. Therefore, when treating such problems, we have to allow for a larger freedom in the choice of the suitable VMO/BMO-like definition (see [3]).

Consider $F_r \in L^{2+\epsilon}(C, \mathbb{R}^{nN})$ is a vector field, (see [8]), a weak solution to system (1.4) is a map

$$u_r \in C^0\big((0,T); L^2(\Omega, \mathbb{R}^N)\big) \cap L^{2+\epsilon}\big(0,T; W^{1,2+\epsilon}(\Omega, \mathbb{R}^N)\big)$$

such that for every $0 < t_1 < t_2 < T$,

$$\sum_r \int_\Omega u_r \varphi(x_r, t_r) dx_r \Big|_{t_1}^{t_2} - \sum_r \int_\Omega \int_{t_1}^{t_2} u_r \varphi_t + a_r(z_r) \langle |Du_r|^{\epsilon} Du_r, D\varphi \rangle \, dz_r$$

$$= -\sum_r \int_\Omega \int_{t_1}^{t_2} \langle |F_r|^{\epsilon} F_r, D\varphi \rangle \, dz_r$$

for every test function $\varphi \in W^{1,2}_{\text{loc}}\big((0,T); L^2(\Omega, \mathbb{R}^N)\big) \cap W^{2+\epsilon}_{\text{loc}}\Big(0,T; W_0^{1,2+\epsilon}(\Omega, \mathbb{R}^N)\Big)$. When dealing with weak solutions, we shall always adopt the formulation via Steklov averages (see again [8]).

Theorem 2.1 Let $u_r \in C^0\big((0,T); L^2(\Omega, \mathbb{R}^N)\big) \cap L^{2+\epsilon}\big(0,T; W^{1,2+\epsilon}(\Omega, \mathbb{R}^N)\big)$ be a weak solution to (1.4), where

$$\epsilon > \frac{-4}{n+2} \tag{2.4}$$

and the function $a_r : C \to \mathbb{R}$ satisfies (2.1) and is strongly VMO. Assume that $|F_r|^{2+\epsilon} \in L^{1+2\epsilon}_{\text{loc}}(C)$ for some $\epsilon > 0$. Then $|Du_r|^{2+\epsilon} \in L^{1+2\epsilon}_{\text{loc}}(C)$. Moreover there exists a constant $c_n \equiv c_n(n, N, \epsilon, \omega(\cdot)) > 1$ such that if $Q_{2R} \Subset C$ then

$$\sum_r \Big(\fint_{Q_R} |Du_r|^{(2+\epsilon)(1+2\epsilon)} \, dz_r \Big)^{\frac{1}{1+2\epsilon}}$$

$$\leq c_n \sum_r \left[\fint_{Q_{2R}} |Du_r|^{2+\epsilon} \, dz_r + \Big(\fint_{Q_{2R}} |F_r|^{(2+\epsilon)(1+2\epsilon)} \, dz_r + 1 \Big)^{\frac{1}{1+2\epsilon}} \right]^d, \tag{2.5}$$

where

$$1 \leq d := \begin{cases} \dfrac{2+\epsilon}{2} & \text{if } \epsilon \geq 0, \\ \dfrac{2(2+\epsilon)}{(2+\epsilon)(n+2) - 2n} & \text{if } \epsilon < 0. \end{cases} \tag{2.6}$$

We also have a result concerning coefficients $a_r(z_r)$ which are not necessarily VMO, but rather have suitably small BMO seminoarm:

Theorem 2.2 Let $u_r \in C^0\big((0,T); L^2(\Omega, \mathbb{R}^N)\big) \cap L^{2+\epsilon}\big(0,T; W^{1,2+\epsilon}(\Omega, \mathbb{R}^N)\big)$ be a weak solution to (1.4), where the function $a_r : C \to \mathbb{R}$ satisfies (2.1) and (2.4). Assume that $|F_r|^{2+\epsilon} \in L^{1+2\epsilon}_{\text{loc}}(C)$ then there exists a number $\epsilon > 0$ such that if $[a_r]_{BMO} \leq \epsilon$, then $|Du_r|^{2+\epsilon} \in L^{1+2\epsilon}_{\text{loc}}(C)$. Moreover, there exists a constant $c_n \equiv c_n(n, N, \epsilon) > 1$ such that (2.5) holds for every $Q_{2R} \Subset C$, with d as in (2.6).

3. Preliminary material

We start with the higher integrability result, see [16], which is essential here in treating the case where the coefficient function $a_r(z_r)$ is not continuous. The version reported here is adapted to our setting from the more general right-hand-side structure in [16] (for the equivalence, see (5.1) with $f \equiv h_1$ in the notation of [16]). Also, the authors of [16] assert that (3.1) holds for some $\delta_0 > 0$, but the fact that it then holds for all $\delta < \delta_0$ may be deduced from their proof; indeed, the only condition on δ_0 is that it must be small.

Theorem 3.1 Let $u_r \in C^0\big((0,T);L^2(\Omega,\mathbb{R}^N)\big) \cap L^{2+\epsilon}\big(0,T;W^{1,2+\epsilon}(\Omega,\mathbb{R}^N)\big)$ be a weak solution to (1.4), where (2.4) is in force and the function $a_r: C \to \mathbb{R}$ satisfies (2.1). Assume that $|F_r|^{2+\epsilon} \in L^{1+2\epsilon}_{\mathrm{loc}}(C)$ for some $\epsilon > 0$. Then there exists $\delta_0 \equiv \delta_0(n,N,\epsilon)$ with $0 < \delta_0 < 2\epsilon(2+\epsilon)$ such that $|Du_r| \in L^{2+\epsilon+\delta}_{\mathrm{loc}}(C)$ for every $0 < \delta \le \delta_0$. Moreover, there exists a constant $c_n \equiv c_n(n,\epsilon) > 1$ such that if $Q_{2R}(R^{2+\epsilon},R) \Subset C$ then

$$\sum_r \Big(⨍_{Q_{z_0}(R^{2+\epsilon}/2^{2+\epsilon},R/2)} |Du_r|^{2+\epsilon+\delta}\,dz_r\Big)^{1/(2+\epsilon+\delta)} \le c_n R^{\sigma(2+\epsilon)-1} \sum_r \Big(⨍_{Q_{z_0}(R^{2+\epsilon},R)} |Du_r|^{2+\epsilon}\,dz_r\Big)^{\sigma} + \frac{c_n}{R} + c_n \sum_r \Big(⨍_{Q_{z_0}(R^{2+\epsilon},R)} |F_r|^{2+\epsilon+\delta}\,dz_r\Big)^{1/(2+\epsilon+\delta)}, \tag{3.1}$$

where

$$\sigma := \frac{2+\delta}{2(2+\epsilon+\delta)}. \tag{3.2}$$

In the remainder of the article, we eventually take $\delta < \delta_0$ in order to have

$$s := 2+\epsilon+\delta \le \min\left\{\frac{2+\epsilon+(2+\epsilon)(1+2\epsilon)}{2}, 3+\epsilon\right\} < (2+\epsilon)(1+2\epsilon), \tag{3.3}$$

and we notice that this implies

$$\frac{(2+\epsilon)(1+2\epsilon)}{(2+\epsilon)(1+2\epsilon)-s} \le \frac{2(1+2\epsilon)}{2\epsilon}. \tag{3.4}$$

The first two lemmas are a consequence of the fundamental L^∞-gradient estimates of [8] and [9].

Lemma 3.1 Let $v_r \in C^0\big((t_1,t_2);L^2(\mathrm{A},\mathbb{R}^N)\big) \cap L^{2+\epsilon}\big(t_1,t_2;W^{1,2+\epsilon}(\mathrm{A},\mathbb{R}^N)\big)$ be a weak solution to

$$(v_r)_t - div(\tilde{a}_r|Dv_r|^{\epsilon}Dv_r) = 0 \quad \text{in} \quad A \times [t_1,t_2), \tag{3.5}$$

where $A \subset \mathbb{R}^n$ is an open set, $t_1 < t_2$, $\epsilon \ge 0$, and $1+\epsilon \le \tilde{a}_r \le 1+2\epsilon$. Assume

$$\sum_r ⨍_{Q(\lambda^{-\epsilon}\varrho^2,\varrho)} |Dv_r|^{2+\epsilon}\,dz_r \le c_{n+1}\lambda^{2+\epsilon}. \tag{3.6}$$

for some $\lambda > 0$ and some cylinder $Q(\lambda^{-\epsilon}\varrho^2,\ \varrho) \Subset A \times [t_1,t_2)$, where c_{n+1} is a given positive constant. Then there exists a constant $c_n > 0$, depending only on n, N, ϵ and c_{n+1}, such that

$$\sum_r \sup_{Q((1/2)\lambda^{-\epsilon}\varrho^2,(1/2)\varrho)} |Dv_r| \le c_n\lambda\,. \tag{3.7}$$

Proof In [8], take $\sigma = 3/4$, we have that if $Q(\theta,\gamma) \Subset A\times[t_1,t_2)$ is a non-degenerate cylinder, then

$$\sum_r \sup_{Q(\theta/2,\gamma/2)} |Dv_r| \le c_n(n,N,\epsilon)\sqrt{\frac{\theta}{\gamma^2}}\sum_r\left(⨍_{Q(\theta,\gamma)}|Dv_r|^{2+\epsilon}\,dz_r\right)^{1/2} + \left(\frac{\gamma^2}{\theta}\right)^{1/\epsilon}. \tag{3.8}$$

Then we take $\theta = \lambda^{-\epsilon}\gamma^2$ and $\gamma = \varrho$, so that $\sqrt{\theta/\gamma^2} = \lambda^{-\epsilon/2}$. Using this fact in the previous inequality, and finally using (3.6), we immediately obtain (3.7). □

In the case $\epsilon < 0$, the estimate that one is allowed to use is different, so we need another statement (and another proof, albeit very similar).

Lemma 3.2 Let $v_r \in C^0\big((t_1,t_2);L^2(\mathrm{A},\mathbb{R}^N)\big)\cap L^{2+\epsilon}\big(t_1,t_2;W^{1,2+\epsilon}(\mathrm{A},\mathbb{R}^N)\big)$ be a weak solution to (3.5), where $A\subset\mathbb{R}^n$ is an open set, $t_1<t_2$, $1+\epsilon\le\tilde{a}\le 1+2\epsilon$, and $\epsilon<0$ satisfies (2.4). Assume

$$\sum_r ⨍_{Q(\varrho^2,\lambda^{\epsilon/2}\varrho)}|Dv_r|^{2+\epsilon}\,dz_r \le c_{n+1}\lambda^{2+\epsilon}. \tag{3.9}$$

for some $\lambda>0$ and some cylinder $Q(\varrho^2,\lambda^{\epsilon/2}\varrho)\Subset A\times[t_1,t_2)$, where c_{n+1} is a given positive constant. Then there exists a constant a $c_n>0$, depending only on , n,N,ϵ and c_{n+1}, such that

$$\sum_r \sup_{Q((1/2)\varrho^2,(1/2)\lambda^{\epsilon/2}\varrho)} |Dv_r| \le c_n\lambda\,. \tag{3.10}$$

Proof This time we use [8] where we can take $r = 2+\epsilon$, since $2+\epsilon$ is assumed to satisfy (2.4). Again taking $\sigma = 3/4$, we have that if $Q(\theta,\gamma)\Subset A\times[t_1,t_2)$ is a non-degenerate cylinder, then

$$\sum_r \sup_{Q(\theta/2,\gamma/2)} |Dv_r| \le c_n(n,N,\epsilon)\left(\frac{\gamma^2}{\theta}\right)^{\frac{n}{[(2+\epsilon)(n+2)-2n]}}\cdot\sum_r\left(⨍_{Q(\theta,\gamma)}|Dv_r|^{2+\epsilon}\,dz_r\right)^{\frac{2}{[(2+\epsilon)(n+2)-2n]}} + \\ + \left(\frac{\gamma^2}{\theta}\right)^{1/-\epsilon}. \tag{3.11}$$

Then we take $\gamma = \lambda^{\epsilon/2}\varrho$ and $\theta = \varrho^2$, so that $\sqrt{\gamma^2/\theta} = \lambda^{\epsilon/2}$. Using this fact in the previous inequality, and finally using (3.9), we immediately obtain (3.10). □

The following two lemmas show how solutions to (1.4) satisfy real reverse-Hölder inequalities when considered on cylinders built according to the intrinsic geometry.

Lemma 3.3 Let $u_r \in C^0\big((0,T);L^2(\Omega,\mathbb{R}^N)\big)\cap L^{2+\epsilon}\big(0,T;W^{1,2+\epsilon}(\Omega,\mathbb{R}^N)\big)$ be a weak solution to (1.4), where $\epsilon\ge 0$, and the function $a_r\colon C\to\mathbb{R}$ satisfies (2.1). Assume that

$$\sum_r\left(⨍_{Q(\lambda^{-\epsilon}\varrho^2,\varrho)}|Du_r|^{2+\epsilon}\,dz_r\right)^{1/(2+\epsilon)} \le c_{n+1}\lambda \tag{3.12}$$

and

$$\lambda \le c_{n+2}\sum_r\left(⨍_{Q(\lambda^{-\epsilon}\varrho^2,\varrho)}|Du_r|^{2+\epsilon}\,dz_r\right)^{1/(2+\epsilon)} + c_{n+2}\sum_r\left(⨍_{Q(\lambda^{-\epsilon}\varrho^2,\varrho)}M^s|F_r|^s\,dz_r\right)^{1/s} \tag{3.13}$$

hold for some $\lambda > 0$ and some cylinder $Q(\lambda^{-\epsilon}\varrho^2, \varrho) \Subset C$, where s is defined in (3.3) via Theorem 3.1, c_{n+1} and c_{n+2} are two given positive constants and $M \geq 1$. Then there exists a constant $c_{n+3} \equiv c_{n+3}(n, N, \epsilon, c_{n+1}, c_{n+2})$ such that

$$\sum_r \left(\fint_{Q((1/2^{2+\epsilon})\lambda^{-\epsilon}\varrho^2,(1/2)\varrho)} |Du_r|^s \, dz_r \right)^{1/s} \leq c_{n+3} \sum_r \left(\fint_{Q(\lambda^{-\epsilon}\varrho^2,\varrho)} |Du_r|^{2+\epsilon} \, dz_r \right)^{1/(2+\epsilon)}$$
$$+ c_{n+3} \sum_r \left(\fint_{Q(\lambda^{-\epsilon}\varrho^2,\varrho)} 1 + M^s |F_r|^s \, dz_r \right)^{1/s}, \qquad (3.14)$$

Proof Without loss of generality we may assume that the cylinder $Q(\lambda^{-\epsilon}\varrho^2, \varrho)$ is centered at the origin. Let us consider the rescaled maps

$$\tilde{u}_r(x_r, t_r) := \frac{u_r(\varrho x_r, \lambda^{-\epsilon}\varrho^2 t_r)}{\varrho\lambda}, \qquad \tilde{F}_r(x_r, t_r) := \frac{F_r(\varrho x_r, \lambda^{-\epsilon}\varrho^2 t_r)}{\lambda},$$

with $(x_r, t_r) \in Q_1$, and the rescaled coefficients $a_r(x_r, t_r) := a_r(\varrho x_r, \lambda^{-\epsilon}\varrho^2 t_r)$. It is easy to check that $\tilde{u}_r \in C^0\big((0,1); L^2(B_1, \mathbb{R}^N)\big) \cap L^{2+\epsilon}\big(0,1; W^{1,2+\epsilon}(B_1, \mathbb{R}^N)\big)$ is a weak solution to the system

$$(\tilde{u}_r)_t - div(\tilde{a}_r(z_r)|D\tilde{u}_r|^\epsilon D\tilde{u}_r) = div\big(|\tilde{F}_r|^\epsilon \tilde{F}_r\big) \quad in \ Q_1.$$

Therefore we may apply Theorem 3.1 and, in particular estimate (3.1) in order to get that there exists a constant c_n, only depending on n, N, ϵ, such that

$$\sum_r \left(\fint_{Q(1/2^{2+\epsilon},1/2)} |D\tilde{u}_r|^s \, dz_r \right)^{1/s} \leq c_n \sum_r \left[\left(\fint_{Q_1} |D\tilde{u}_r|^{2+\epsilon} \, dz_r \right)^{\sigma} + \left(\fint_{Q_1} |\tilde{F}_r|^s \, dz_r \right)^{1/s} \right] + c_n, \quad (3.15)$$

where, according to (3.2), $\sigma = (-\epsilon + s)/2s$. Scaling back in (3.15) yields

$$\sum_r \left(\fint_{Q((1/2^{2+\epsilon})\lambda^{-\epsilon}\varrho^2,(1/2)\varrho)} |Du_r|^s \, dz_r \right)^{1/s} \leq c_n \lambda^{1-\sigma(2+\epsilon)} \sum_r \left(\fint_{Q(\lambda^{-\epsilon}\varrho^2,\varrho)} |Du_r|^{2+\epsilon} \, dz_r \right)^{\sigma}$$
$$+ c_n \sum_r \left(\fint_{Q(\lambda^{-\epsilon}\varrho^2,\varrho)} |F_r|^s \, dz_r \right)^{1/s} + c_n\lambda. \quad (3.16)$$

Here $c_n \equiv c_n(n, N, \epsilon)$. But using (3.12) and (3.13) we have

$$\lambda^{1-\sigma(2+\epsilon)} \sum_r \left(\fint_{Q(\lambda^{-\epsilon}\varrho^2,\varrho)} |Du_r|^{2+\epsilon} \, dz_r \right)^{\sigma} \leq c_n\lambda \leq c_n \sum_r \left(\fint_{Q(\lambda^{-\epsilon}\varrho^2,\varrho)} |Du_r|^{2+\epsilon} \, dz_r \right)^{1/2+\epsilon}$$
$$+ c_n \sum_r \left(\fint_{Q(\lambda^{-\epsilon}\varrho^2,\varrho)} M^s |F_r|^s \, dz_r \right)^{1/s},$$

where $c_n \equiv c_n(c_{n+1}, c_{n+2})$. Finally, (3.14) follows, connecting the last inequalities to (3.16). □

Lemma 3.4 Let $u_r \in C^0\big((0,T); L^2(\Omega, \mathbb{R}^N)\big) \cap L^{2+\epsilon}\big(0,T; W^{1,2+\epsilon}(\Omega, \mathbb{R}^N)\big)$ be a weak solution to (1.4), where $\frac{-4}{n+2} < \epsilon \leq 0$, and the function $a_r: C \to \mathbb{R}$ satisfies (2.1). Assume

$$\sum_r \left(\fint_{Q(\varrho^2, \lambda^{\epsilon/2}\varrho)} |Du_r|^{2+\epsilon} \, dz_r \right)^{1/(2+\epsilon)} \leq c_{n+1}\lambda$$

and

$$\lambda \le c_{n+2}\sum_r \left(⨍_{Q(\varrho^2,\lambda^{\epsilon/2}\varrho)} |Du_r|^{2+\epsilon}\,dz_r\right)^{1/(2+\epsilon)} + c_{n+2}\sum_r \left(⨍_{Q(\varrho^2,\lambda^{\epsilon/2}\varrho)} M^s|F_r|^s\,dz_r\right)^{1/s}$$

hold for some $\lambda > 0$ and for some cylinder $Q(\varrho^2, \lambda^{\epsilon/2}\varrho) \Subset C$, where s is defined in (3.3), c_{n+1} and c_{n+2} are two given positive constants and $M \ge 1$. Then there exists a constant $c_{n+3} \equiv c_{n+3}(n, N, \epsilon, c_{n+1}, c_{n+2})$ such that

$$\sum_r \left(⨍_{Q((1/2^{2+\epsilon})\varrho^2,(1/2)\lambda^{\epsilon/2}\varrho)} |Du_r|^s\,dz_r\right)^{1/s} \le c_{n+3}\lambda^{1-\sigma(2+\epsilon)}\sum_r \left(⨍_{Q(\varrho^2,\lambda^{\epsilon/2}\varrho)} |Du_r|^{2+\epsilon}\,dz_r\right)^{1/(2+\epsilon)}$$
$$+c_{n+3}\sum_r \left(⨍_{Q(\varrho^2,\lambda^{\epsilon/2}\varrho)} (1 + M^s|F_r|^s)dz_r\right)^{1/s}$$

Proof Again we assume that $Q(\varrho^2, \lambda^{\epsilon/2}\varrho)$ is centered at the origin. This time, we consider the rescaled maps

$$\tilde{u}_r(x_r, t_r) := \frac{u_r\left(\lambda^{\epsilon/2}\varrho x_r, \varrho^2 t_r\right)}{\varrho\lambda^{(2+\epsilon)/2}}, \qquad \tilde{F}_r(x_r, t_r) := \frac{F_r\left(\lambda^{\epsilon/2}\varrho x_r, \varrho^2 t_r\right)}{\lambda},$$

with $(x_r, t_r) \in Q_1$, and the re-scaled coefficients $\tilde{u}(x_r, t_r) := a_r(\lambda^{\epsilon/2}\varrho x_r, \varrho^2 t_r)$. The remainder of the proof now follows exactly as in the previous Lemma 3.3. □

We conclude this section with a couple of elementary results: the first can be promptly adapted from [7]; the second can be found in [8], with slight modifications.

Lemma 3.5 Let $\epsilon > 0$ and $\mu_r \in [0,1]$; there exists a constant $c_n \equiv c_n(n, N, \epsilon)$ such that if $v_r, w_r \in \mathbb{R}^{nN}$ then:

$$(\mu_r^2 + |A|^2)^{(1+\epsilon)/2} \le c_n(\mu_r^2 + |B|^2)^{(1+\epsilon)/2} + c_n(\mu_r^2 + |B|^2 + |A|^2)^{(-1+\epsilon)/2}|B - A|^2.$$

Lemma 3.6 Let $0 < \epsilon < \infty$ and $\mu_r \in [0,1]$. There exists a constant $c_n \equiv c_n(n, N, \epsilon)$, independent of μ_r, such that for any $A, B \in \mathbb{R}^{nN}$

$$(\mu_r^2 + |B|^2 + |A|^2)^{(-1+\epsilon)/2}|B - A|^2 \le c_n\langle(\mu_r^2 + |B|^2)^{(-1+\epsilon)/2}B - (\mu_r^2 + |A|^2)^{(-1+\epsilon)/2}A, B - A\rangle.$$

When $\mu_r = |A| = |B| = 0$, and $\epsilon < 1$, the quantities involved in the previous inequality are meant to be zero.

4. Proof of Theorems 2.1 and 2.2

By an approximation argument, in Step 1 we reduce the proof of Theorem 2.1 to proving (2.5) when the solution has locally bounded gradient. Then we devote the remaining steps to this last task. Theorem 2.2 then follows easily.

1.4 Proof of Theorem 2.1. Step 1: Approximation. We first show how to approximate the solution u_r of (1.4), in neighborhood of a given cylinder, with a sequence $(u_r)_\epsilon$ of solutions to similar problems, whose gradients are bounded. Let $Q_{2R} \equiv (t_0 - (2R)^2, t_0 + (2R)^2 \times B_{x_0}(2R)) \Subset C$ be as in the statement of Theorem 2.1, and let $Q_{2\bar{R}} \Subset C$ be a cylinder, concentric with Q_{2R}, with $\bar{R} > R$. Let $\phi_1 : \mathbb{R}^n \to \mathbb{R}$, and $\phi_2 : \mathbb{R} \to \mathbb{R}$ be two standard mollifiers with compact support in B^1 and $(-1,1)$, respectively, and for all $\epsilon < (1/2)dist(Q_{2\bar{R}}, \partial C)$ and $(x_r, t_r) \in Q_{2\bar{R}}$ define

$$(F_r)_\epsilon(x_r, t_r) := \int_{Q_1}\sum_r F_r(x_r + \epsilon y_r, t_r + \epsilon s_r)\phi_1(y_r)\phi_2(s_r)dy_r ds_r$$

and

$$(a_r)_\epsilon(x_r,t_r) := \int_{Q_1}\sum_r a_r(x_r+\epsilon y_r, t_r+\epsilon s_r)\phi_1(y_r)\phi_2(s_r)dy_r ds_r$$

Clearly, $(F_r)_\epsilon \in C^\infty(Q_{2\bar{R}},\mathbb{R}^{nN})$, and $(a_r)_\epsilon \in C^\infty(Q_{2\bar{R}})$. Moreover,

$$(F_r)_\epsilon \to F_r \quad \text{strongly in } L^{(2+\epsilon)(1+2\epsilon)}(Q_{2\bar{R}},\mathbb{R}^{nN}), \tag{4.1}$$

$$(a_r)_\epsilon \to a_r \quad \text{strongly in } L^{t_r}(Q_{2\bar{R}},\mathbb{R}^{nN}), \qquad \forall t_r<\infty \tag{4.2}$$

Finally, the new functions $(a_r)_\epsilon$ satisfy (2.1). Now we define the map

$$(u_r)_\epsilon \in C^0\left((t_0-(2\bar{R})^2, t_0+(2\bar{R})^2); L^2\left(B_{x_0}(\bar{R}),\mathbb{R}^N\right)\right) \cap L^{2+\epsilon}\left(t_0-(2\bar{R})^2, t_0+(2\bar{R})^2; W^{1,2+\epsilon}(B_{x_0}(\bar{R}),\mathbb{R}^N)\right)$$

as the unique solution to the following Cauchy-Dirichlet problem:

$$\begin{cases} ((u_r)_\epsilon)_t - div((a_r)_\epsilon(z_r)|D(u_r)_\epsilon|^\epsilon D(u_r)_\epsilon) = \mathrm{div}(|(F_r)_\epsilon|^\epsilon (F_r)_\epsilon), & \text{in } Q_{2\bar{R}} \\ (u_r)_\epsilon \equiv u_r & \text{on } \partial_{2+\epsilon}Q_{2\bar{R}} \end{cases} \tag{4.3}$$

The existence of such $(u_r)_\epsilon$ follows from the theory of monotone operators, or via Galerkin approximation (see [21]); for such problems and their exact meaning, (see [8]). Our aim is now to show that

$$D(u_r)_\epsilon \to Du_r \quad \text{strongly in } L^{2+\epsilon}(Q_{2\bar{R}},\mathbb{R}^{nN}), \tag{4.4}$$

Using the fact that both u_r and $(u_r)_\epsilon$ are weak solutions, we have that

$$((u_r)_\epsilon - u_r)_t - div\big((a_r)_\epsilon(z_r)(|D(u_r)_\epsilon|^\epsilon D(u_r)_\epsilon - |Du_r|^\epsilon Du_r)\big) = div((a_r)_\epsilon(z_r) - a_r(z_r)|Du_r|^\epsilon Du_r), + div(|(F_r)_\epsilon|^\epsilon (F_r)_\epsilon - |F_r|^\epsilon F_r F_r).$$

Now we test the previous identity with the map $(u_r)_\epsilon - u_r$, which is possible modulo Steklov averages (for the definition, see [8]); note that this is an admissible test map since $u_r \equiv (u_r)_\epsilon$ on $\partial_{2+\epsilon}Q_{2\bar{R}}$. After asimple computation we arrive at

$$\sup_{(t_0-(2\bar{R})^2 \le t \le t_0+(2\bar{R})^2)} \int_{B_{x_0}(2\bar{R})}\sum_r |(u_r)_\epsilon(x_r-t_r) - u_r(x_r-t_r)|^2 dx_r + \int_{Q_{2\bar{R}}}\sum_r (a_r)_\epsilon(z_r)\langle |D(u_r)_\epsilon|^\epsilon D(u_r)_\epsilon - |Du_r|^\epsilon Du_r\, , D(u_r)_\epsilon - Du_r\rangle\, dz_r$$

$$\le c_n\left|\int_{Q_{2\bar{R}}}\sum_r (a_r)_\epsilon(z_r) - a_r(z_r)\langle |Du_r|^\epsilon Du_r\, , D(u_r)_\epsilon - Du_r\rangle\, dz_r\right| + c_n\left|\int_{Q_{2\bar{R}}}\sum_r \langle |(F_r)_\epsilon|^\epsilon (F_r)_\epsilon - |F_r|^\epsilon F_r\, , D(u_r)_\epsilon - Du_r\rangle\, dz_r\right|, \tag{4.5}$$

and therefore, using (2.1),

$$\int_{Q_{2\bar{R}}}\sum_r |D(u_r)_\epsilon|^{2+\epsilon}\, dz_r$$

$$\leq c_n \int_{Q_{2\bar{R}}} \sum_r [|(a_r)_\epsilon(z_r)| + |a_r(z_r)|](|Du_r|^{1+\epsilon}|D(u_r)_\epsilon| + |D(u_r)_\epsilon|^{1+\epsilon}|Du_r| + |Du_r|^{2+\epsilon})\, dz_r$$

$$+c_n \int_{Q_{2\bar{R}}} \sum_r (|(F_r)_\epsilon| + |F_r|)^{1+\epsilon}(|D(u_r)_\epsilon| + |Du_r|)\, dz_r.$$

Finally using Young's inequality in a standard way, and the definitions of $(F_r)_\epsilon$ and $(a_r)_\epsilon$, we get

$$\int_{Q_{2\bar{R}}} \sum_r |D(u_r)_\epsilon|^{2+\epsilon}\, dz_r \leq c_n \int_C \sum_r (|Du_r|^{2+\epsilon} + |F_r|^{2+\epsilon})\, dz_r \leq c_{n+1}. \tag{4.6}$$

Now we go back to (4.5). In the following, we use the expression

$$(A, B) \mapsto (|A|^2 + |B|^2)^{\epsilon/2}|B - A|^2, \qquad A, B \in \mathbb{R}^{nN},$$

which is already defined in Lemma 3.6 and involves a singularity when $|A| = |B| = 0$ and $\epsilon < 0$. In this case, the meaning of the previous quantity was defined as zero. Using Lemma 3.6 with $\mu_r = 0$, together with (2.1), we find

$$\int_{Q_{2\bar{R}}} \sum_r (|D(u_r)_\epsilon|^2 + |Du_r|^2)^{\epsilon/2}|D(u_r)_\epsilon - Du_r|^2\, dz_r$$

$$\leq c_n(n, N, \epsilon) \int_{Q_{2\bar{R}}} \sum_r (a_r)_\epsilon(z_r)\langle |D(u_r)_\epsilon|^\epsilon D(u_r)_\epsilon - |Du_r|^\epsilon Du_r\, , D(u_r)_\epsilon - Du_r\rangle\, dz_r,$$

and from (4.5) we have

$$\int_{Q_{2\bar{R}}} \sum_r (|D(u_r)_\epsilon|^2 + |Du_r|^2)^{\epsilon/2}|D(u_r)_\epsilon - Du_r|^2\, dz_r$$

$$\leq c_n \int_{Q_{2\bar{R}}} \sum_r |(a_r)_\epsilon(z_r) - a_r(z_r)||Du_r|^{1+\epsilon}|D(u_r)_\epsilon - Du_r|\, dz_r,$$

$$+c_n \int_{Q_{2\bar{R}}} \sum_r |\,|(F_r)_\epsilon|^\epsilon (F_r)_\epsilon - |F_r|^\epsilon F_r|\,|D(u_r)_\epsilon - Du_r|\, dz_r, \tag{4.7}$$

with $c_n \equiv c_n(n, N, \epsilon)$. Using Young's inequality with $\delta \in (0,1)$ we find

$$\int_{Q_{2\bar{R}}} \sum_r (|D(u_r)_\epsilon|^2 + |Du_r|^2)^{\epsilon/2}|D(u_r)_\epsilon - Du_r|^2\, dz_r$$

$$\leq c_n(\delta) \int_{Q_{2\bar{R}}} \sum_r |(a_r)_\epsilon(z_r) - a_r(z_r)|^{\frac{2+\epsilon}{1+\epsilon}}|Du_r|^{2+\epsilon}\, dz_r + \delta \int_{Q_{2\bar{R}}} \sum_r |D(u_r)_\epsilon|^{2+\epsilon} + |Du_r|^{2+\epsilon}\, dz_r$$

$$+c_n(\delta) \int_{Q_{2\bar{R}}} \sum_r ||(F_r)_\epsilon|^\epsilon (F_r)_\epsilon - |F_r|^\epsilon F_r|^{\frac{2+\epsilon}{1+\epsilon}}\, dz_r, \tag{4.8}$$

with the constant c_n depending also on n, N, ϵ. We have now, recalling that s is the higher integrability exponent defined in (3.3), we have

$$\int_{Q_{2\bar{R}}} \sum_r |(a_r)_\epsilon(z_r) - a_r(z_r)|^{\frac{2+\epsilon}{1+\epsilon}}|Du_r|^{2+\epsilon}\, dz_r$$

$$\leq \left(\int_{Q_{2\bar{R}}} \sum_r |(a_r)_\epsilon(z_r) - a_r(z_r)|^{\frac{(2+\epsilon)s}{(1+\epsilon)(s-2-\epsilon)}} \, dz_r \right)^{\frac{(s-2-\epsilon)}{s}} \cdot \left(\int_{Q_{2\bar{R}}} \sum_r |Du_r|^s \, dz_r \right)^{\frac{2+\epsilon}{s}} \xrightarrow{(4.2)} 0 \quad (4.9)$$

as $\epsilon \longrightarrow 0$. By a dominated convergence argument and (4.1) we directly have

$$\int_{Q_{2\bar{R}}} \sum_r | \, |(F_r)_\epsilon|^\epsilon (F_r)_\epsilon - |F_r|^\epsilon F_r \, |^{(2+\epsilon)/(1+\epsilon)} \, dz_r \longrightarrow 0 \quad (4.10)$$

as $\epsilon \longrightarrow 0$. taking into account (4.6), connecting (4.8) – (4.10), and finally letting $\delta \longrightarrow 0$, we obtain

$$\lim_{\epsilon \longrightarrow 0} \int_{Q_{2\bar{R}}} \sum_r (|D(u_r)_\epsilon|^2 + |Du_r|^2)^{\epsilon/2} |D(u_r)_\epsilon - Du_r|^2 \, dz_r = 0. \quad (4.11)$$

Now, if $\epsilon < 0$, using Hölder inequality and again (4.6)

$$\int_{Q_{2\bar{R}}} \sum_r |D(u_r)_\epsilon - Du_r|^{2+\epsilon} \, dz_r \leq \left(\int_{Q_{2\bar{R}}} \sum_r |D(u_r)_\epsilon|^{2+\epsilon} + |Du_r|^{2+\epsilon} \, dz_r \right)^{1/2} \cdot$$

$$\left(\int_{Q_{2\bar{R}}} \sum_r (|D(u_r)_\epsilon|^2 + |Du_r|^2)^{\epsilon/2} |D(u_r)_\epsilon - Du_r|^2 \, dz_r \right)^{1/2};$$

therefore

$$\lim_{\epsilon \longrightarrow 0} \int_{Q_{2\bar{R}}} \sum_r |D(u_r)_\epsilon - Du_r|^{2+\epsilon} \, dz_r = 0. \quad (4.12)$$

which proves (4.4) in the case $\epsilon < 0$. If instead $\epsilon \geq 0$, going back to (4.7) and using Young's inequality we get

$$\int_{Q_{2\bar{R}}} \sum_r |D(u_r)_\epsilon - Du_r|^{2+\epsilon} \, dz_r \leq \int_{Q_{2\bar{R}}} \sum_r (|D(u_r)_\epsilon|^2 + |Du_r|^2)^{\frac{\epsilon}{2}} |D(u_r)_\epsilon - Du_r|^2 \, dz_r,$$

and (4.12) follows from (4.11). Now we finally show how the validity of (2.5) in the general case follows from the case when Du_r is bounded. Therefore, let us assume that (2.5) holds whenever Du_r is bounded. We consider the maps $\{(u_r)_\epsilon\}$ defined in (4.3); the regularity theory for the parabolic $(2+\epsilon)$-Laplacian systems applies, see [8, Chapter 8], and therefore $D(u_r)_\epsilon \in L^\infty(Q_{2R}, \mathbb{R}^{nN})$. Then by (2.5)

$$\left(\fint_{Q_R} \sum_r |Du_r|^{(2+\epsilon)(1+2\epsilon)} \, dz_r \right)^{1/(1+2\epsilon)} \leq \liminf_{\epsilon \longrightarrow 0} \left(\fint_{Q_R} \sum_r |D(u_r)_\epsilon|^{(2+\epsilon)(1+2\epsilon)} \, dz_r \right)^{1/(1+2\epsilon)}$$

$$\leq c_n \lim_{\epsilon \longrightarrow 0} \left[\fint_{Q_{2R}} \sum_r |D(u_r)_\epsilon|^{2+\epsilon} \, dz_r + \left(\fint_{Q_{2R}} \sum_r |(F_r)_\epsilon|^{(2+\epsilon)(1+2\epsilon)} \, dz_r + 1 \right)^{1/(1+2\epsilon)} \right]^d$$

$$= c_n \left[\fint_{Q_{2R}} \sum_r |Du_r|^{2+\epsilon} \, dz_r + \left(\fint_{Q_{2R}} \sum_r |F_r|^{(2+\epsilon)(1+2\epsilon)} \, dz_r + 1 \right)^{1/(1+2\epsilon)} \right]^d,$$

where we used (4.4) and Fatou's lemma to manage for the left-hand side. The remainder of the proof is therefore dedicated to proving (2.5) under the additional assumption that Du_r is bounded. Once (2.5) is

proved in the general case, the full statement $|Du_r|^{(2+\epsilon)} \in L^{1+2\epsilon}_{\text{loc}}(C)$ follows via a standard covering argument.

Step 2: A stopping time argument. We start with the case $\epsilon \geq 0$. We define

$\lambda_0 > 1$ according to

$$\lambda_0^{1/d} := \left(\fint_{Q_{2R}} \sum_r |Du_r|^{2+\epsilon} \, dz_r \right)^{1/(2+\epsilon)} + \left(\fint_{Q_{2R}} \sum_r M^s |F_r|^s \, dz_r \right)^{1/s} + 1, \tag{4.13}$$

where number d was defined in (2.6). The number $M > 1$ will be chosen later, in a universal way, that will only depend on the fixed parameters n, N, ϵ. Now pick any two numbers γ, λ such that

$$\frac{R}{2^{8(2+\epsilon)}} \leq \gamma \leq \frac{R}{2}, \qquad B\lambda_0 := 2^{10(n+2)t_r} \lambda_0 \leq \lambda. \tag{4.14}$$

We check that for all $z_0 \in Q_R$,

$$\left(\fint_{Q_{z_0}(\lambda^{-\epsilon}\gamma^2,\gamma)} \sum_r |Du_r|^{2+\epsilon} \, dz_r \right)^{1/(2+\epsilon)} + \left(\fint_{Q_{z_0}(\lambda^{-\epsilon}\gamma^2,\gamma)} \sum_r M^s |F_r|^s \, dz_r \right)^{1/s} < \lambda. \tag{4.15}$$

Indeed, we first remark that if $z_0 \in Q_R$, and $\varrho < R$, since $\lambda > \lambda_0 > 1$ and $\epsilon \geq 0$, then $Q_z(\lambda^{-\epsilon}\varrho^2, \varrho) \subset Q_{2R}$, so that, in particular, when γ satisfies (4.14),

$$\frac{Q((2R)^2, 2R)}{Q_{z_0}(\lambda^{-\epsilon}\gamma^2, \gamma)} > 1. \tag{4.16}$$

Then

$$\begin{aligned}
&\left(\fint_{Q_{z_0}(\lambda^{-\epsilon}\gamma^2,\gamma)} \sum_r |Du_r|^{2+\epsilon} \, dz_r \right)^{1/(2+\epsilon)} + \left(\fint_{Q_{z_0}(\lambda^{-\epsilon}\gamma^2,\gamma)} \sum_r M^s |F_r|^s \, dz_r \right)^{1/s} \\
&\leq \left(\frac{|Q((2R)^2, 2R)|}{|Q_{z_0}(\lambda^{-\epsilon}\gamma^2, \gamma)|} \right)^{1/(2+\epsilon)} \cdot \left(\fint_{Q_{2R}} \sum_r |Du_r|^{2+\epsilon} \, dz_r \right)^{1/(2+\epsilon)} \\
&\quad + \left(\frac{|Q((2R)^2, 2R)|}{|Q_{z_0}(\lambda^{-\epsilon}\gamma^2, \gamma)|} \right)^{1/s} \cdot \left(\fint_{Q_{2R}} \sum_r M^s |F_r|^s \, dz_r \right)^{1/s} \\
&\overset{(3.3),(4.13),(4.14),(4.16)}{<} \left(2^{10(n+2)(2+\epsilon)}\lambda^{\epsilon}\right)^{1/(2+\epsilon)} \lambda_0^{1/d} \\
&\overset{(2.6)}{=} \left(2^{10(n+2)(2+\epsilon)}\lambda^{\epsilon}\right)^{1/(2+\epsilon)} \lambda_0^{2/(2+\epsilon)} \overset{(4.14)}{\leq} \lambda.
\end{aligned}$$

Now, with λ as in (4.14), take a point $z_0 \in Q_R$, such that $|Du_r(z_0)| > \lambda$. By Lebesgue's differentiation theorem, for almost every such point we have

$$\lim_{\varrho \to 0} \left\{ \left(\fint_{Q_{z_0}(\lambda^{-\epsilon}\gamma^2,\gamma)} \sum_r |Du_r|^{2+\epsilon} \, dz_r \right)^{\frac{1}{(2+\epsilon)}} + \left(\fint_{Q_{z_0}(\lambda^{-\epsilon}\gamma^2,\gamma)} \sum_r M^s |F_r|^s \, dz_r \right)^{\frac{1}{s}} \right\} > \lambda. \tag{4.17}$$

Assume that for some $\varrho > 0$ satisfying $\varrho \leq R/2$

$$\left(⨍_{Q_{z_0}(\lambda^{-\epsilon}\varrho^2,\varrho)}\sum_r |Du_r|^{2+\epsilon}\,dz_r\right)^{\frac{1}{(2+\epsilon)}}+\left(⨍_{Q_{z_0}(\lambda^{-\epsilon}\varrho^2,\varrho)}\sum_r M^s|F_r|^s\,dz_r\right)^{\frac{1}{s}}>\lambda.$$

We note that some such ϱ exist by (4.17); since for $\varrho \geq R/2^{8(2+\epsilon)}$, the opposite inequality holds by (4.14), (4.15), necessarily we conclude that $\varrho < R/2^{8(2+\epsilon)}$. Therefore we can select a radius $\varrho_{z_0} \leq R/2$ to be the largest for which

$$\left(⨍_{Q_{z_0}(\lambda^{-\epsilon}\varrho_{z_0}^2,\varrho_{z_0})}\sum_r |Du_r|^{2+\epsilon}\,dz_r\right)^{\frac{1}{(2+\epsilon)}}+\left(⨍_{Q_{z_0}(\lambda^{-\epsilon}\varrho_{z_0}^2,\varrho_{z_0})}\sum_r M^s|F_r|^s\,dz_r\right)^{\frac{1}{s}}=\lambda, \quad (4.18)$$

in the sense that if $\frac{R}{2} \geq \varrho > \varrho_{z_0}$, then

$$\left(⨍_{Q_{z_0}(\lambda^{-\epsilon}\varrho^2,\varrho)}\sum_r |Du_r|^{2+\epsilon}\,dz_r\right)^{\frac{1}{(2+\epsilon)}}+\left(⨍_{Q_{z_0}(\lambda^{-\epsilon}\varrho^2,\varrho)}\sum_r M^s|F_r|^s\,dz_r\right)^{\frac{1}{s}}<\lambda.$$

By this argumentation, it must be

$$\varrho_{z_0} < \frac{R}{2^{8(2+\epsilon)}}. \quad (4.19)$$

Since $\lambda > 1$, and $\epsilon \geq 0$, we immediately have that

$$Q_{z_0}\left(\lambda^{-\epsilon}\left(2^{j(2+\epsilon)}\varrho_{z_0}\right)^2,\left(2^{j(2+\epsilon)}\varrho_{z_0}\right)\right) \subset Q\left((2R)^2,(2R)\right), \qquad j \in \{0,\dots,5\}.$$

Moreover we observe that for $j \in \{0,\dots,5\}$ we have

$$\frac{\lambda}{8^{j(2+\epsilon)}} \leq \left(⨍_{Q_{z_0}(\lambda^{-\epsilon}(2^{j(2+\epsilon)}\varrho_{z_0})^2,(2^{j(2+\epsilon)}\varrho_{z_0}))}\sum_r |Du_r|^{2+\epsilon}\,dz_r\right)^{1/(2+\epsilon)}$$
$$+\left(⨍_{Q_{z_0}(\lambda^{-\epsilon}(2^{j(2+\epsilon)}\varrho_{z_0})^2,(2^{j(2+\epsilon)}\varrho_{z_0}))}\sum_r M^s|F_r|^s\,dz_r\right)^{1/s} \leq \lambda. \quad (4.20)$$

Indeed, the right-hand-side inequality just follows from the choice of ϱ_{z_0}, while as for the left-hand side, the sum of the integrals appearing in (4.20) can be estimated from below as follows:

$$\left(⨍_{Q_{z_0}(\lambda^{-\epsilon}(2^{j(2+\epsilon)}\varrho_{z_0})^2,(2^{j(2+\epsilon)}\varrho_{z_0}))}\sum_r |Du_r|^{2+\epsilon}\,dz_r\right)^{1/(2+\epsilon)}$$
$$+\left(⨍_{Q_{z_0}(\lambda^{-\epsilon}(2^{j(2+\epsilon)}\varrho_{z_0})^2,(2^{j(2+\epsilon)}\varrho_{z_0}))}\sum_r M^s|F_r|^s\,dz_r\right)^{1/s}$$
$$\geq \left(\frac{\left|Q_{z_0}(\lambda^{-\epsilon}\varrho_{z_0}^2,\ \varrho_{z_0})\right|}{\left|Q_{z_0}(\lambda^{-\epsilon}(2^{j(2+\epsilon)}\varrho_{z_0})^2,(2^{j(2+\epsilon)}\varrho_{z_0}))\right|}\right)^{1/(2+\epsilon)}\cdot\left(⨍_{Q_{z_0}(\lambda^{-\epsilon}\varrho_{z_0}^2,\varrho_{z_0})}\sum_r |Du_r|^{2+\epsilon}\,dz_r\right)^{1/(2+\epsilon)}$$
$$+\left(⨍_{Q_{z_0}(\lambda^{-\epsilon}\varrho_{z_0}^2,\varrho_{z_0})}\sum_r M^s|F_r|^s\,dz_r\right)^{1/s}$$

$$= \frac{\lambda}{8^{j(2+\epsilon)}}. \tag{4.18}$$

Now let us consider the level set

$$E(\lambda) := \{z_r \in Q_R : |Du_r(z_r)| > \lambda\}.$$

For a.e. $z_0 \in E(\lambda)$ we can find a cube $Q_{z_0}(\lambda^{-\epsilon}\varrho_{z_0}^2, \varrho_{z_0}) \subset Q_{2R}$ as constructed above and, in particular, such that (4.20) holds for $j \in \{0, \dots, 5\}$. Therefore, applying Vitali's covering theorem, we find a family of disjoint cubes $\{Q_i^0\}$ of the type considered up to now:

$$Q_i^0 \equiv Q_{z_i}(\lambda^{-\epsilon}\varrho_{z_i}^2, \varrho_{z_i}) \subset Q_{2R}, \qquad z_i \in E(\lambda), \tag{4.21}$$

such that

$$E(\lambda) \subset \bigcup_{i\in\mathbb{N}} Q_i^1 \cup \text{negligible set.}$$

Here, we have denoted

$$Q_i^1 \equiv Q_{z_i}\left(\lambda^{-\epsilon}\left(2^{3(2+\epsilon)}\varrho_{z_i}\right)^2, \left(2^{3(2+\epsilon)}\varrho_{z_i}\right)\right).$$

For future convenience, we also introduce

$$Q_i^2 \equiv Q_{z_i}\left(\lambda^{-\epsilon}\left(2^{4(2+\epsilon)}\varrho_{z_i}\right)^2, \left(2^{4(2+\epsilon)}\varrho_{z_i}\right)\right).$$

and

$$Q_i^3 \equiv Q_{z_i}\left(\lambda^{-\epsilon}\left(2^{5(2+\epsilon)}\varrho_{z_i}\right)^2, \left(2^{5(2+\epsilon)}\varrho_{z_i}\right)\right).$$

We now deal with the case $\epsilon < 0$. The basic change with respect to the case $\epsilon \geq 0$, and following the subquadratic scaling, (see [8]), is to use cubes of the type $Q_z(\varrho^2, \lambda^{\epsilon/2}\varrho)$. In this case, λ_0 is still defined as in (4.13), and γ, λ are again picked according to (4.14). With $z_0 \in Q_R$, once again, we have

$$\left(⨍_{Q_{z_0}(\gamma^2,\lambda^{\epsilon/2}\gamma)} \sum_r |Du_r|^{2+\epsilon}\, dz_r\right)^{1/(2+\epsilon)} + \left(⨍_{Q_{z_0}(\gamma^2,\lambda^{\epsilon/2}\gamma)} \sum_r M^s|F_r|^s\, dz_r\right)^{1/s} < \lambda. \tag{4.22}$$

The equivalent of (4.16) in this case is

$$\left(\frac{|Q((2R)^2, 2R)|}{|Q_{z_0}(\gamma^2, \lambda^{\epsilon/2}\gamma)|}\right) > 1. \tag{4.23}$$

Then we have

$$\left(⨍_{Q_{z_0}(\gamma^2,\lambda^{\epsilon/2}\gamma)} \sum_r |Du_r|^{2+\epsilon}\, dz_r\right)^{1/(2+\epsilon)} + \left(⨍_{Q_{z_0}(\gamma^2,\lambda^{\epsilon/2}\gamma)} \sum_r M^s|F_r|^s\, dz_r\right)^{1/s}$$

$$\leq \left(\frac{|Q((2R)^2, 2R)|}{\left|Q_{z_0}\left(\gamma^2, \lambda^{\frac{\epsilon}{2}}\gamma\right)\right|}\right)^{1/(2+\epsilon)} \cdot \left(⨍_{Q_{2R}} \sum_r |Du_r|^{2+\epsilon}\, dz_r\right)^{1/(2+\epsilon)}$$

$$+\left(\frac{|Q((2R)^2,2R)|}{\left|Q_{z_0}\left(\gamma^2,\lambda^{\frac{\epsilon}{2}}\gamma\right)\right|}\right)^{1/s}\cdot\left(⨍_{Q_{2R}}\sum_r M^s|F_r|^s\,dz_r\right)^{1/s}$$

$$\overset{(3.3),(4.13),(4.14),(4.23)}{\leq}\left(2^{10(n+2)(2+\epsilon)}\lambda^{-n\epsilon/2}\right)^{1/(2+\epsilon)}\lambda_0^{1/d}$$

$$\overset{(2.6)}{=}\left(2^{10(n+2)(2+\epsilon)}\lambda^{-n\epsilon/2}\right)^{1/(2+\epsilon)}\lambda_0^{((2+\epsilon)(n+2)-2n)/2}$$

$$\overset{(2.4),(4.14)}{\leq}\lambda,$$

and (4.22) is proved. An important remark to be made is that, beside needing it for Lemmas 3.2 and 3.4 and Kinnunen and Lewis's theorem, (see Theorem 3), this is the only point where we need (2.4). For the remainder, we can proceed exactly as for the case $\epsilon \geq 0$ but using the cubes of the type $Q_{z_i}(\varrho_{z_i}^2,\lambda^{\epsilon/2}\varrho_{z_i}) \subset Q_{2R}$, instead of those of the type $Q_{z_i}\left(\lambda^{-\epsilon}\varrho_{z_i}^2,\varrho_{z_i}\right) \subset Q_{2R}$. At the end, we come up with a family of disjoint cubes $\{Q_i^0\}$ of the type

$$Q_i^0 \equiv Q_{z_i}(\varrho_{z_i}^2,\lambda^{\epsilon/2}\varrho_{z_i}) \subset Q_{2R}, \qquad z_i \in E(\lambda), \tag{4.24}$$

such that (4.19) holds and having the fundamental property that for $j \in \{0,\dots,5\}$ and all i,

$$\frac{\lambda}{8^{j(2+\epsilon)}} \leq \left(⨍_{Q_{z_i}((2^{j(2+\epsilon)}\varrho_{z_i})^2,\lambda^{\epsilon/2}(2^{j(2+\epsilon)}\varrho_{z_i}))}\sum_r |Du_r|^{2+\epsilon}\,dz_r\right)^{1/(2+\epsilon)}$$

$$+\left(⨍_{Q_{z_i}((2^{j(2+\epsilon)}\varrho_{z_i})^2,\lambda^{\epsilon/2}(2^{j(2+\epsilon)}\varrho_{z_i}))}\sum_r M^s|F_r|^s\,dz_r\right)^{1/s} \leq \lambda. \tag{4.25}$$

and such that

$$E(\lambda) \subset \bigcup_{i\in\mathbb{N}} Q_i^1 \cup \text{negligible set.}$$

Accordingly, in the case $\epsilon < 0$, we are denoting

$$Q_i^1 \equiv Q_{z_i}\left(\left(2^{3(2+\epsilon)}\varrho_{z_i}\right)^2,\lambda^{\epsilon/2}\left(2^{3(2+\epsilon)}\varrho_{z_i}\right)\right).$$

$$Q_i^2 \equiv Q_{z_i}\left(\left(2^{4(2+\epsilon)}\varrho_{z_i}\right)^2,\lambda^{\epsilon/2}\left(2^{4(2+\epsilon)}\varrho_{z_i}\right)\right).$$

and

$$Q_i^3 \equiv Q_{z_i}\left(\left(2^{5(2+\epsilon)}\varrho_{z_i}\right)^2,\lambda^{\epsilon/2}\left(2^{5(2+\epsilon)}\varrho_{z_i}\right)\right).$$

From now on, for the remainder of the proof, when dealing with cubes of the type $Q_i^0,\dots,Q_i^3$, we implicitly understand which kind we are using, depending on ϵ.

Step 3: Comparison maps. When $\epsilon \geq 0$, on the cube Q_i^2 centered at $(z_r)_i := ((x_r)_i,(t_r)_i)$, we define the map

$$(v_r)_i \in C^0\left(\left((t_r)_i - 2^{8(2+\epsilon)}\lambda^{-\epsilon}\varrho_i^2,(t_r)_i + 2^{8(2+\epsilon)}\lambda^{-\epsilon}\varrho_i^2\right);L^2\left(B_{(x_r)_i}2^{4(2+\epsilon)}\varrho_i\right),\mathbb{R}^N\right)$$

$$\cap L^{2+\epsilon}\left((t_r)_i - 2^{8(2+\epsilon)}\lambda^{-\epsilon}\varrho_i^2, (t_r)_i + 2^{8(2+\epsilon)}\lambda^{-\epsilon}\varrho_i^2); W^{1,2+\epsilon}\left(B_{(x_r)_i}2^{4(2+\epsilon)}\varrho_i\right), \mathbb{R}^N\right)$$

as the unique solution to the Cauchy-Dirichlet problem

$$\begin{cases} ((v_r)_i)_t - div((a_r)_i|D(v_r)_i|^\epsilon D(v_r)_i) = 0 & \text{in } Q_i^2, \\ (v_r)_i \equiv u_r & \text{on } \partial_{2+\epsilon}Q_i^2 \end{cases} \tag{4.26}$$

(see again [8, pages 20 – 2, 296]), where

$$(a_r)_i := \fint_{Q_i^2} \sum_r a_r(z_r)\, dz_r.$$

We note that, due to (2.1)

$$1 + \epsilon \le (a_r)_i \le 1 + 2\epsilon. \tag{4.27}$$

We now find some estimates on $(v_r)_i$. Using the fact that both u_r and $(v_r)_i$ are solutions, we have

$$(u_r - (v_r)_i)_t - div\big((a_r)_i(|Du_r|^\epsilon Du_r - |D(v_r)_i|^\epsilon D(v_r)_i)\big)$$
$$= div\big((a_r(z_r) - (a_r)_i)|Du_r|^\epsilon Du_r\big) + div(|F_r|^\epsilon DF_r),$$

in the weak sense. Now we shall proceed formally, as in Step1, by testing the previous equality with the map $\phi_r = u_r - (v_r)_i$, which may be justified via Steklov averages. Again, it is crucial that u_r and $(v_r)_i$ agree on the parabolic boundary $\partial_{2+\epsilon}Q_i^2$. As in Step1, we obtain the equivalent of (4.5),

$$\int_{Q_i^2}\sum_r (a_r)_i\langle |Du_r|^\epsilon Du_r - |D(v_r)_i|^\epsilon D(v_r)_i\,, Du_r - D(v_r)_i\rangle\, dz_r$$
$$\le c_n\left|\int_{Q_i^2}\sum_r ((a_r)_i(z_r) - a_r(z_r))\langle |Du_r|^\epsilon Du_r\,, Du_r - D(v_r)_i\rangle\, dz_r\right|$$
$$+c_n\left|\int_{Q_i^2}\sum_r \langle |F_r|^\epsilon F_r\,, Du_r - D(v_r)_i\rangle\, dz_r\right|, \tag{4.28}$$

and, using (4.27), the equivalent of (4.6),

$$\int_{Q_i^2}\sum_r |D(v_r)_i|^{2+\epsilon}\, dz_r \le c_n(n, N, \epsilon)\int_{Q_i^2}\sum_r (|Du_r|^{2+\epsilon} + |F_r|^{2+\epsilon})\, dz_r, \tag{4.29}$$

Therefore, recalling that $M \ge 1$ and using Hölder inequality, we find

$$\int_{Q_i^2}\sum_r |D(v_r)_i|^{2+\epsilon}\, dz_r \le c_n\int_{Q_i^2}\sum_r |Du_r|^{2+\epsilon}\, dz_r + c_n\left(\int_{Q_i^2}\sum_r M^s|F_r|^s\, dz_r\right)^{\frac{2+\epsilon}{s}} \overset{(4.20)}{\le} c_n\lambda^{2+\epsilon}, \tag{4.30}$$

where $c_n \equiv c_n(n, N, \epsilon)$. Now remark that

$$Q_i^1 \subset Q_{z_i}\left(\frac{\lambda^{-\epsilon}(2^{4(2+\epsilon)}\varrho_i)^2}{2}, \frac{(2^{4(2+\epsilon)}\varrho_i)}{2}\right);$$

therefore, by (4.30) we can apply Lemma 3.1, with $Q \equiv Q_i^2$, to get that there exists an absolute constant A_1, depending only on n, N, ϵ, such that

$$A_1 \geq 1, \qquad \sup_{Q_i^1} \sum_r |D(v_r)_i| \leq A_1 \lambda. \tag{4.31}$$

When $\epsilon < 0$, we can proceed in a completely analogous way, invoking Lemma 3.2 instead of Lemma 3.1 and using the right kind of cubes, and (4.31) follows again. We note in particular that (4.28) – (4.31) hold both for $\epsilon \geq 0$ and $\epsilon < 0$.

Now we want to get an estimate for the integral

$$\int_{Q_i^2} \sum_r (|Du_r|^2 + |D(v_r)_i|^2)^{\epsilon/2} \, |Du_r - D(v_r)_i|^2 dz_r.$$

Using Lemma 3.6 we have

$$\begin{aligned}
&\frac{1+\epsilon}{c_n(n,N,2+\epsilon)} \int_{Q_i^2} \sum_r (|Du_r|^2 + |D(v_r)_i|^2)^{\epsilon/2} \, |Du_r - D(v_r)_i|^2 dz_r \\
&\overset{(4.27)}{\leq} (a_r)_i \int_{Q_i^2} \sum_r \langle |Du_r|^\epsilon Du_r - |D(v_r)_i|^\epsilon D(v_r)_i \, , Du_r - D(v_r)_i \rangle dz_r \\
&\overset{(4.28)}{\leq} c_n \int_{Q_i^2} \sum_r |a_r(z_r) - (a_r)_i| |Du_r|^{1+\epsilon} |Du_r - D(v_r)_i| dz_r \\
&\quad + c_n \int_{Q_i^2} \sum_r |F_r|^{1+\epsilon} |Du_r - D(v_r)_i| dz_r.
\end{aligned} \tag{4.32}$$

Using Young's inequality, with $\delta \in (0,1)$ we have

$$\begin{aligned}
&\int_{Q_i^2} \sum_r |a_r(z_r) - (a_r)_i| |Du_r|^{1+\epsilon} |Du_r - D(v_r)_i| dz_r \\
&\leq \frac{c_n}{\delta^{\frac{1}{1+\epsilon}}} \int_{Q_i^2} \sum_r |a_r(z_r) - (a_r)_i|^{\frac{2+\epsilon}{1+\epsilon}} |Du_r|^{2+\epsilon} dz_r + \delta \int_{Q_i^2} \sum_r (|Du_r|^{2+\epsilon} + |D(v_r)_i|^{2+\epsilon}) \, dz_r \\
&\overset{(4.29)}{\leq} \frac{c_n}{\delta^{\frac{1}{1+\epsilon}}} \int_{Q_i^2} \sum_r |a_r(z_r) - (a_r)_i|^{\frac{2+\epsilon}{1+\epsilon}} |Du_r|^{2+\epsilon} dz_r + c_n \delta \int_{Q_i^2} \sum_r (|Du_r|^{2+\epsilon} + |D(v_r)_i|^{2+\epsilon}) \, dz_r
\end{aligned} \tag{4.33}$$

and

$$\begin{aligned}
&\int_{Q_i^2} \sum_r |F_r|^{1+\epsilon} |Du_r - D(v_r)_i| dz_r \\
&\leq \frac{c_n}{\delta^{1/(1+\epsilon)}} \int_{Q_i^2} \sum_r |F_r|^{2+\epsilon} dz_r + \delta \int_{Q_i^2} \sum_r (|Du_r|^{2+\epsilon} + |D(v_r)_i|^{2+\epsilon}) \, dz_r \\
&\overset{(4.29)}{\leq} \frac{c_n}{\delta^{1/(1+\epsilon)}} \int_{Q_i^2} \sum_r |F_r|^{2+\epsilon} dz_r + c_n \delta \int_{Q_i^2} \sum_r |Du_r|^{2+\epsilon} \, dz_r
\end{aligned}$$

with $c_n \equiv c_n(n,N,\epsilon)$. Connecting the last two inequalities with (4.32), we finally have

$$\int_{Q_i^2} \sum_r (|Du_r|^2 + |D(v_r)_i|^2)^{\epsilon/2} |Du_r - D(v_r)_i|^2 \, dz_r$$

$$\leq \frac{c_n}{\delta^{1/(1+\epsilon)}} \int_{Q_i^2} \sum_r |a_r(z_r) - (a_r)_i|^{(2+\epsilon)/(1+\epsilon)} |Du_r|^{2+\epsilon}\, dz_r$$

$$+ \frac{c_n}{\delta^{1/(1+\epsilon)}} \int_{Q_i^2} \sum_r |F_r|^{2+\epsilon} dz_r + c_n \delta \int_{Q_i^2} \sum_r |Du_r|^{2+\epsilon}\, dz_r \tag{4.34}$$

with $c_n \equiv c_n(n, N, \epsilon)$, and $\delta \in (0,1)$ not yet chosen.

We estimate the first integral appearing in the right-hand side of (4.34). Using Hölder inequality we have

$$\int_{Q_i^2} \sum_r |a_r(z_r) - (a_r)_i|^{(2+\epsilon)/(1+\epsilon)} |Du_r|^{2+\epsilon}\, dz_r$$

$$\leq \left(\fint_{Q_i^2} \sum_r |a_r(z_r) - (a_r)_i|^b\, dz_r \right)^{(s-2-\epsilon)/s} \left(\fint_{Q_i^2} \sum_r |Du_r|^s\, dz_r \right)^{(2+\epsilon)/s} |Q_i^2|,$$

where we have set

$$b := \frac{2+\epsilon}{1+\epsilon} \frac{s}{s-2-\epsilon} > 1.$$

We remark that

$$\left(\fint_{Q_i^2} \sum_r |a_r(z_r) - (a_r)_i|^b\, dz_r \right)^{(s-2-\epsilon)/s} \leq (2L)^{(b-1)(s-2-\epsilon)/s} [\omega(R)]^{(s-2-\epsilon)/s}$$

as a consequence of (2.1), (2.3), (4.19), (4.27), while

$$\left(\fint_{Q_i^2} \sum_r |Du_r|^s\, dz_r \right)^{(2+\epsilon)/s} \leq c_n \fint_{Q_i^3} \sum_r |Du_r|^{2+\epsilon}\, dz_r + c_n \left(\fint_{Q_i^3} \sum_r (1 + M^s |F_r|^s)\, dz_r \right)^{(2+\epsilon)/s}$$

as a consequence of Lemma 3.3 or Lemma 3.4. Merging the last three estimates with (4.34) we finally obtain the estimate we were looking for:

$$\int_{Q_i^2} \sum_r (|Du_r|^2 + |D(v_r)_i|^2)^{\epsilon/2} |Du_r - D(v_r)_i|^2\, dz_r \leq c_n \left\{ \frac{[\omega(R)]^{\frac{(s-2-\epsilon)}{s}}}{\delta^{1/(1+\epsilon)}} + \delta \right\} \fint_{Q_i^3} \sum_r |Du_r|^{2+\epsilon}\, dz_r$$

$$+ \frac{c_n [\omega(R)]^{\frac{(s-2-\epsilon)}{s}}}{\delta^{1/(1+\epsilon)}} \left(\fint_{Q_i^3} \sum_r (1 + M^s |F_r|^s)\, dz_r \right)^{(2+\epsilon)/s} |Q_i^2| + \frac{c_n}{\delta^{1/(1+\epsilon)}} \fint_{Q_i^3} \sum_r |F_r|^{2+\epsilon}\, dz_r \tag{4.35}$$

where the constant c_n depends on the data n, N, ϵ, we estimated $|Q_i^3| \leq 4^{10n(2+\epsilon)} |Q_i^0|$ and $\delta \in (0,1)$ is yet to be chosen.

Step 4: Estimates on cubes. Lemma 3.5 with $\mu_r = 0$ implies

$$|Du_r|^{2+\epsilon} \leq c_{n+l} |D(v_r)_i|^{2+\epsilon} + c_{n+l} (|Du_r|^2 + |D(v_r)_i|^2)^{\epsilon/2} |Du_r - D(v_r)_i|^2$$

where $c_{n+l} \equiv c_{n+l}(n, \epsilon)$ is the constant appearing in the lemma. Accordingly, we fix the constant

$$A := (1 + 2c_{n+l}) A_1, \tag{4.36}$$

where A_1 is the constant appearing in (4.31). In this way, A depends only on the data n, N, ϵ. We have

$$\begin{aligned}
&\left|\{z_r \in Q_i^1 : |Du_r|^{2+\epsilon} > A\lambda^{2+\epsilon}\}\right| \\
&\quad \le \left|\{z_r \in Q_i^1 : (|Du_r|^2 + |D(v_r)_i|^2)^{\epsilon/2}|Du_r - D(v_r)_i|^2 > A_1\lambda^{2+\epsilon}\}\right| \\
&\quad + \left|\{z_r \in Q_i^1 : |D(v_r)_i|^{2+\epsilon} > A_1\lambda^{2+\epsilon}\}\right| \quad \left(\overset{(4.31),\,(4.36)}{=} 0\right) \\
&\quad = \left|\{z_r \in Q_i^1 : (|Du_r|^2 + |D(v_r)_i|^2)^{\epsilon/2}|Du_r - D(v_r)_i|^2 > A_1\lambda^{2+\epsilon}\}\right|,
\end{aligned}$$

so that

$$\left|\{z_r \in Q_i^1 : |Du_r|^{2+\epsilon} > A\lambda^{2+\epsilon}\}\right| \le \frac{1}{A_1\lambda^{2+\epsilon}} \int_{Q_i^1} \sum_r (|Du_r|^2 + |D(v_r)_i|^2)^{\epsilon/2}|Du_r - D(v_r)_i|^2 \, dx_r,$$

and using (4.35), (4.36)

$$\left|\{z_r \in Q_i^1 : |Du_r|^{2+\epsilon} > A\lambda^{2+\epsilon}\}\right| \le \frac{c_n}{A\lambda^{2+\epsilon}} \left\{ \frac{[\omega(R)]^{(s-2-\epsilon)/s}}{\delta^{1/(1+\epsilon)}} + \delta \right\} \int_{Q_i^3} \sum_r |Du_r|^{2+\epsilon} \, dz_r$$

$$+ \frac{c_n[\omega(R)]^{(s-2-\epsilon)/s}}{A\lambda^{2+\epsilon}\delta^{1/(1+\epsilon)}} \left(\fint_{Q_i^3} \sum_r (1 + M^s|F_r|^s) \, dz_r \right)^{\frac{2+\epsilon}{s}} \left|Q_i^0\right| + \frac{c_n}{A\lambda^{2+\epsilon}\delta^{1/(1+\epsilon)}} \int_{Q_i^3} \sum_r |F_r|^{2+\epsilon} \, dz_r. \quad (4.37)$$

We now carefully estimate these three integrals; we note that the constant c_n just seen depends only on n, ϵ. Since we will later backtrack to find the exact dependence on ϵ; we will be careful to let every constant c_n be independent of q; given (3.3), when not essential, we majorize constants as, for example, 2^s by $c_n = c_n(\epsilon)$. We first provide an estimate for $\left|Q_i^0\right|$; by (4.18), (4.21) or the analogous expressions for $\epsilon < 0$, either of the following inequalities must be true:

$$\left(\frac{\lambda}{2}\right)^{2+\epsilon} \le \frac{1}{\left|Q_i^0\right|} \int_{Q_i^0} \sum_r |Du_r|^{2+\epsilon} \, dz_r \qquad \text{or} \qquad \left(\frac{\lambda}{2}\right)^{s} \le \frac{1}{\left|Q_i^0\right|} \int_{Q_i^0} \sum_r M^s|F_r|^s \, dz_r \, .$$

In any case,

$$\left|Q_i^0\right| \le \frac{2^{2+\epsilon}}{\lambda^{2+\epsilon}} \int_{Q_i^0} \sum_r |Du_r|^{2+\epsilon} \, dz_r + \frac{2^s}{\lambda^s} \int_{Q_i^0} \sum_r M^s|F_r|^s \, dz_r. \quad (4.38)$$

We now split the last integral as follows: for some $\gamma > 0$,

$$\begin{aligned}
\frac{1}{\lambda^s} \int_{Q_i^0} \sum_r M^s|F_r|^s \, dz_r &= \frac{1}{\lambda^s} \int_{Q_i^0 \cap \{|F_r| > \gamma\lambda\}} \sum_r M^s|F_r|^s \, dz_r + \frac{1}{\lambda^s} \int_{Q_i^0 \cap \{|F_r| \le \gamma\lambda\}} \sum_r M^s|F_r|^s \, dz_r \\
&\le \frac{1}{\lambda^s} \int_{Q_i^0 \cap \{|F_r| > \gamma\lambda\}} \sum_r M^s|F_r|^s \, dz_r + M^s\gamma^s\left|Q_i^0\right|. \quad (4.39)
\end{aligned}$$

Choosing

$$\gamma^s := \frac{1}{2^{s+1}M^s}, \quad (4.40)$$

connecting (4.39) to (4.38), and reabsorbing $|Q_i^0|/2$, we find the estimate for $|Q_i^0|$ in which we are interested :

$$|Q_i^0| \le \frac{2^{3+\epsilon}}{\lambda^{2+\epsilon}} \int_{Q_i^0} \sum_r |Du_r|^{2+\epsilon}\, dz_r + \frac{2^{s+1}}{\lambda^s} \int_{Q_i^0 \cap \{|F_r| > \gamma\lambda\}} \sum_r M^s |F_r|^s\, dz_r. \tag{4.41}$$

Now we gain a further estimate, again splitting with some $\tau > 0$:

$$\frac{1}{\lambda^{2+\epsilon}} \int_{Q_i^0} \sum_r |Du_r|^{2+\epsilon}\, dz_r = \frac{1}{\lambda^{2+\epsilon}} \int_{Q_i^0 \cap \{|Du_r| > \tau\lambda\}} \sum_r |Du_r|^{2+\epsilon}\, dz_r + \frac{1}{\lambda^{2+\epsilon}} \int_{Q_i^0 \cap \{|Du_r| \le \tau\lambda\}} \sum_r |Du_r|^{2+\epsilon}\, dz_r$$

$$\le \frac{1}{\lambda^{2+\epsilon}} \int_{Q_i^0 \cap \{|Du_r| > \tau\lambda\}} \sum_r |Du_r|^{2+\epsilon}\, dz_r + \tau^{2+\epsilon} |Q_i^0|$$

$$\overset{(4.41)}{\le} \frac{1}{\lambda^{2+\epsilon}} \int_{Q_i^0 \cap \{|Du_r| > \tau\lambda\}} \sum_r |Du_r|^{2+\epsilon}\, dz_r + \frac{2^{3+\epsilon}\tau^{2+\epsilon}}{\lambda^{2+\epsilon}} \int_{Q_i^0} \sum_r |Du_r|^{2+\epsilon}\, dz_r$$

$$+ \frac{2^{s+1}\tau^{2+\epsilon}}{\lambda^s} \int_{Q_i^0 \cap \{|F_r| > \gamma\lambda\}} \sum_r M^s |F_r|^s\, dz_r. \tag{4.42}$$

Choosing

$$\tau^{2+\epsilon} := \frac{1}{2^{4+\epsilon}} \tag{4.43}$$

and reabsorbing the second-to-last integral into the left-hand side of (4.42), we conclude with $\frac{1}{\lambda^{2+\epsilon}} \int_{Q_i^0} \sum_r |Du_r|^{2+\epsilon}\, dz_r \le \frac{2}{\lambda^{2+\epsilon}} \int_{Q_i^0 \cap \{|Du_r| > \tau\lambda\}} \sum_r |Du_r|^{2+\epsilon}\, dz_r + \frac{2^{s-2-\epsilon}}{\lambda^s} \int_{Q_i^0 \cap \{|F_r| > \gamma\lambda\}} \sum_r M^s |F_r|^s\, dz_r.$

In particular, from (4.41) we deduce

$$|Q_i^0| \le \frac{2^{4+\epsilon}}{\lambda^{2+\epsilon}} \int_{Q_i^0 \cap \{|Du_r| > \tau\lambda\}} \sum_r |Du_r|^{2+\epsilon}\, dz_r + \frac{2^{s+2}}{\lambda^s} \int_{Q_i^0 \cap \{|F_r| > \gamma\lambda\}} \sum_r M^s |F_r|^s\, dz_r. \tag{4.44}$$

Then we have

$$\frac{1}{\lambda^{2+\epsilon}} \int_{Q_i^3} \sum_r |Du_r|^{2+\epsilon}\, dz_r = \frac{|Q_i^3|}{\lambda^{2+\epsilon}} \fint_{Q_i^3} \sum_r |Du_r|^{2+\epsilon}\, dz_r \overset{(4.20)/(4.25)}{\le} |Q_i^3| \le 2^{5(2+\epsilon)(n+2)} |Q_i^0|$$

$$\overset{(4.44)}{\le} \frac{c_n}{\lambda^{2+\epsilon}} \int_{Q_i^0 \cap \{|Du_r| > \tau\lambda\}} \sum_r |Du_r|^{2+\epsilon}\, dz_r + \frac{c_n}{\lambda^s} \int_{Q_i^0 \cap \{|F_r| > \gamma\lambda\}} \sum_r M^s |F_r|^s\, dz_r. \tag{4.45}$$

(As we said, we omitted a 2^s here). Similarly, using Hölder inequality,

$$\frac{1}{\lambda^{2+\epsilon}} \int_{Q_i^3} \sum_r |F_r|^{2+\epsilon}\, dz_r = \frac{|Q_i^3|}{M^{2+\epsilon}\lambda^{2+\epsilon}} \fint_{Q_i^3} \sum_r M^{2+\epsilon} |F_r|^{2+\epsilon}\, dz_r$$

$$\le \frac{c_n |Q_i^0|}{M^{2+\epsilon}\lambda^{2+\epsilon}} \left(\fint_{Q_i^3} \sum_r M^s |F_r|^s\, dz_r \right)^{(2+\epsilon)/s} \overset{(4.20)/(4.25)}{\le} \frac{c_n |Q_i^0|}{M^{2+\epsilon}}$$

$$\overset{(4.44)}{\leq} \frac{c_n}{M^{2+\epsilon}\lambda^{2+\epsilon}} \int_{Q_i^0 \cap \{|Du_r| > \tau\lambda\}} \sum_r |Du_r|^{2+\epsilon}\, dz_r$$

$$+ \frac{c_n}{M^{2+\epsilon}\lambda^s} \int_{Q_i^0 \cap \{|F_r| > \gamma\lambda\}} \sum_r M^s |F_r|^s\, dz_r. \tag{4.46}$$

Finally, since $\lambda > 1$, we have

$$\frac{|Q_i^0|}{\lambda^{2+\epsilon}} \left(\fint_{Q_i^3} \sum_r (1 + M^s |F_r|^s)\, dz_r \right)^{(2+\epsilon)/s} \overset{(4.20)/(4.25)}{\leq} |Q_i^0|$$

$$\overset{(4.44)}{\leq} \frac{c_n}{\lambda^{2+\epsilon}} \int_{Q_i^0 \cap \{|Du_r| > \tau\lambda\}} \sum_r |Du_r|^{2+\epsilon}\, dz_r + \frac{c_n}{\lambda^s} \int_{Q_i^0 \cap \{|F_r| > \gamma\lambda\}} \sum_r M^s |F_r|^s\, dz_r. \tag{4.47}$$

Connecting (4.45) – (4.47) to (4.37), we have the estimate on the cubes which we were looking for:

$$\left|\{z_r \in Q_i^1 : |Du_r|^{2+\epsilon} > A\lambda^{2+\epsilon}\}\right|$$

$$\leq \frac{c_n}{A\lambda^{2+\epsilon}} \left\{ \frac{[\omega(R)]^{(s-2-\epsilon)/s}}{\delta^{1/(1+\epsilon)}} + \frac{1}{M^{2+\epsilon}\delta^{1/(1+\epsilon)}} + \delta \right\} \int_{Q_i^0 \cap \{|Du_r| > \tau\lambda\}} \sum_r |Du_r|^{2+\epsilon}\, dz_r$$

$$+ \frac{c_n M^s}{A\lambda^s} \left\{ \frac{[\omega(R)]^{(s-2-\epsilon)/s}}{\delta^{1/(1+\epsilon)}} + \frac{1}{M^{2+\epsilon}\delta^{1/(1+\epsilon)}} + \delta \right\} \int_{Q_i^0 \cap \{|F_r| > \gamma\lambda\}} \sum_r |F_r|^s\, dz_r.$$

Define

$$G \equiv G(\delta, M, R) := \frac{[\omega(R)]^{(s-2-\epsilon)/s}}{\delta^{1/(1+\epsilon)}} + \frac{1}{M^{2+\epsilon}\delta^{1/(1+\epsilon)}} + \delta, \tag{4.48}$$

and the estimate above may be rewritten as

$$\left|\{z_r \in Q_i^1 : |Du_r|^{2+\epsilon} > A\lambda^{2+\epsilon}\}\right| \leq G(\delta, M, R) \left\{ \frac{c_n}{A\lambda^{2+\epsilon}} \int_{Q_i^0 \cap \{|Du_r| > \tau\lambda\}} \sum_r |Du_r|^{2+\epsilon}\, dz_r \right.$$

$$\left. + \frac{c_n M^s}{A\lambda^s} \int_{Q_i^0 \cap \{|F_r| > \gamma\lambda\}} \sum_r |F_r|^s\, dz_r \right\}. \tag{4.49}$$

Here the constant c_n only depends on n, N, ϵ, while γ and τ have been chosen in (4..40) and (4.43), respectively. The constant $\delta \in (0,1)$ is not yet chosen.

Step 5: Final estimate. Here, we prove the validity of (2.5), provided

$$R \leq R_0, \tag{4.50}$$

where $R_0 \equiv R_0\big(n, N, \epsilon, \omega(\cdot)\big) > 0$ is a radius we will determine in (4.56) below. The general case follows via a standard covering argument.

Using the fact that the cubes $\{Q_i^0\}$ are disjoint, summing up on $i \in \mathbb{N}$ in (4.49) we have, by (4.21) or (4.24), since $A > 1$ by (4.31), (4.36),

$$\left|\left\{z_r \in Q_R : |Du_r| > A^{1/(2+\epsilon)}\lambda\right\}\right| \leq \sum_i \left|\{z_r \in Q_i^1 : |Du_r|^{2+\epsilon} > A\lambda^{2+\epsilon}\}\right|$$

$$\leq G(\delta,M,R)\left\{\frac{c_n}{A\lambda^{2+\epsilon}}\int_{Q_i^0\cap\{|Du_r|>\tau\lambda\}}\sum_r|Du_r|^{2+\epsilon}\,dz_r+\frac{c_nM^s}{A\lambda^s}\int_{Q_i^0\cap\{|F_r|>\gamma\lambda\}}\sum_r|F_r|^s\,dz_r\right\}.$$

The previous inequality holds for every $\lambda\geq B\lambda_0$ (recall (4.14)). Therefore, integrating with respect to λ yields

$$\int_{B\lambda_0}^{\infty}\lambda^{(2+\epsilon)(1+2\epsilon)-1}\sum_r\left|\{z_r\in Q_R:|Du_r|>A^{1/(2+\epsilon)}\lambda\}\right|d\lambda$$

$$\leq\frac{c_nG(\delta,M,R)}{A}\int_{B\lambda_0}^{\infty}\lambda^{(2+\epsilon)(1+2\epsilon)-(2+\epsilon)-1}\int_{Q_{2R}\cap\{|Du_r|>\tau\lambda\}}\sum_r|Du_r|^{2+\epsilon}\,dz_r\,d\lambda$$

$$\leq\frac{c_nM^sG(\delta,M,R)}{A}\int_{B\lambda_0}^{\infty}\lambda^{(2+\epsilon)(1+2\epsilon)-s-1}\int_{Q_{2R}\cap\{|F_r|>\gamma\lambda\}}\sum_r|F_r|^s\,dz_r\,d\lambda. \tag{4.51}$$

We recall that for any measurable function $g_r\geq 0$ and any $\beta>\alpha>1$,

$$\int\sum_r g_r^{\beta}dx_r=(\beta-\alpha)\int_0^{+\infty}t^{\beta-\alpha-1}\int_{\{x_r:g_r(x_r)>t_r\}}\sum_r g_r^{\alpha}dx_r\,dt\,; \tag{4.52}$$

then we have, using Fubini's theorem in a standard way,

$$\int_{Q_R}\sum_r|Du_r|^{(2+\epsilon)(1+2\epsilon)}\,dz_r$$

$$=(2+\epsilon)(1+2\epsilon)\int_0^{\infty}(A^{1/(2+\epsilon)}\lambda)^{(2+\epsilon)(1+2\epsilon)-1}\sum_r\left|\{z_r\in Q_R:|Du_r|>A^{1/(2+\epsilon)}\lambda\}\right|d(A^{1/(2+\epsilon)}\lambda)$$

$$=(2+\epsilon)(1+2\epsilon)\int_0^{B\lambda_0}(\text{same})d\lambda+(2+\epsilon)(1+2\epsilon)\int_{B\lambda_0}^{\infty}(\text{same})d\lambda$$

$$\leq A^{1+2\epsilon}B^{(2+\epsilon)(1+2\epsilon)}\lambda_0^{(2+\epsilon)(1+2\epsilon)}|Q_R|$$

$$+(2+\epsilon)(1+2\epsilon)A^{1+2\epsilon}\int_{B\lambda_0}^{\infty}\lambda^{(2+\epsilon)(1+2\epsilon)-1}\sum_r\left|\{z_r\in Q_R:|Du_r|>A^{1/(2+\epsilon)}\lambda\}\right|d\lambda$$

$$\overset{(4.36),(4.51)}{\leq} A^{1+2\epsilon}B^{(2+\epsilon)(1+2\epsilon)}\lambda_0^{(2+\epsilon)(1+2\epsilon)}|Q_R|+$$

$$+\frac{c_n(2+\epsilon)(1+2\epsilon)A^{1+2\epsilon}G(\delta,M,R)}{\tau^{(2+\epsilon)(1+2\epsilon)-(2+\epsilon)}}\int_0^{\infty}(\tau\lambda)^{(2+\epsilon)(1+2\epsilon)-(2+\epsilon)-1}\int_{Q_{2R}\cap\{|Du_r|>\tau\lambda\}}\sum_r|Du_r|^{2+\epsilon}\,dz_r\,d(\tau\lambda)$$

$$+\frac{c_n(2+\epsilon)(1+2\epsilon)A^{1+2\epsilon}M^sG(\delta,M,R)}{\gamma^{(2+\epsilon)(1+2\epsilon)-s}}\int_0^{\infty}(\gamma\lambda)^{(2+\epsilon)(1+2\epsilon)-s-1}\int_{Q_{2R}\cap\{|F_r|>\gamma\lambda\}}\sum_r|F_r|^s\,dz_r\,d(\gamma\lambda)$$

$$\overset{(4.40),(4.43),(4.52)}{\leq} A^{1+2\epsilon}B^{(2+\epsilon)(1+2\epsilon)}\lambda_0^{(2+\epsilon)(1+2\epsilon)}|Q_R|$$

$$+c_n\frac{(2+\epsilon)}{2\epsilon}2^{(4+\epsilon)(1+2\epsilon)}A^{1+2\epsilon}G(\delta,M,R)\int_{Q_{2R}}\sum_r|Du_r|^{(2+\epsilon)(1+2\epsilon)}\,dz_r$$

$$+c_n \frac{(2+\epsilon)(1+2\epsilon)}{(2+\epsilon)(1+2\epsilon)-s} A^{1+2\epsilon} 2^{(2+\epsilon)(1+2\epsilon)} M^{(2+\epsilon)(1+2\epsilon)} G(\delta, M, R) \int_{Q_{2R}} \sum_r |F_r|^{(2+\epsilon)(1+2\epsilon)} \, dz_r$$

$$\overset{(3.4)}{\leq} \quad \tilde{c}_n A^{1+2\epsilon} B^{(2+\epsilon)(1+2\epsilon)} \lambda_0^{(2+\epsilon)(1+2\epsilon)} |Q_R|$$

$$+\tilde{c}_n 2^{(4+\epsilon)(1+2\epsilon)} A^{1+2\epsilon} \frac{(2+\epsilon)}{2\epsilon} G(\delta, M, R) \left\{ \int_{Q_{2R}} \sum_r |Du_r|^{(2+\epsilon)(1+2\epsilon)} \, dz_r \right.$$

$$\left. +M^{(2+\epsilon)(1+2\epsilon)} \int_{Q_{2R}} \sum_r |F_r|^{(2+\epsilon)(1+2\epsilon)} \, dz_r \right\}. \tag{4.53}$$

where $\tilde{c}_n \equiv \tilde{c}_n(n, N, \epsilon)$, and the dependence on ϵ is explicitly stated. We note that nothing depends on λ_0 except the first term at the right-hand side.

Now we look again at (4.48), we fix choose ϵ_0 so that it only depends on n, and we first choose $\delta > 0$ small enough in order to have

$$\tilde{c}_n \delta \leq \frac{2\epsilon}{6(1+2\epsilon) A^{1+2\epsilon} 2^{(4+\epsilon)(1+2\epsilon)}} \epsilon_0. \tag{4.54}$$

Since $\tilde{c}_n$ depends only on n, N, ϵ, this also fixes δ as a number depending only on n, N, ϵ. We can therefore select $M \equiv M(\tilde{c}_n, \delta) \equiv M(n, N, \epsilon) > 1$ large enough in order to have (large-M-inequality principle)

$$\frac{\tilde{c}_n}{M^{2+\epsilon}} \leq \frac{2\epsilon\delta^{1/(1+\epsilon)}}{6(1+2\epsilon) A^{1+2\epsilon} 2^{(4+\epsilon)(1+2\epsilon)}} \epsilon_0. \tag{4.55}$$

We may finally select $R_0 \equiv R_0(n, N, \epsilon, \omega(\cdot)) > 0$, small enough to ensure

$$\tilde{c}_n [\omega(R_0)]^{(s-2-\epsilon)/s} \leq \frac{2\epsilon\delta^{1/(1+\epsilon)}}{6(1+2\epsilon) A^{1+2\epsilon} 2^{(4+\epsilon)(1+2\epsilon)}} \epsilon_0. \tag{4.56}$$

Therefore, taking R as we said in (4.50), keeping into account (4.48) and inequalities (4.54) – (4.56), we obtain

$$\tilde{c}_n 2^{(4+\epsilon)(1+2\epsilon)} A^{1+2\epsilon} \frac{1+2\epsilon}{2\epsilon} G(\delta, M, R) \leq \frac{1}{2} \epsilon_0. \tag{4.57}$$

Plugging this last inequality into (4.54) and finally passing to averages, we get, by (4.14),

$$\left(\fint_{Q_R} \sum_r |Du_r|^{(2+\epsilon)(1+2\epsilon)} \, dz_r \right)^{1/(1+2\epsilon)} \leq c_n \lambda_0^{2+\epsilon} + c_n \left(\fint_{Q_{2R}} \sum_r |F_r|^{(2+\epsilon)(1+2\epsilon)} \, dz_r \right)^{1/(1+2\epsilon)}$$

$$\left(\epsilon_0 \fint_{Q_{2R}} \sum_r |Du_r|^{(2+\epsilon)(1+2\epsilon)} \, dz_r \right)^{1/(1+2\epsilon)}, \tag{4.58}$$

where now c_n depends also on $1+2\epsilon$; that is $c_n \equiv c_n(n, N, \epsilon)$. Now we recall the choice of λ_0 in (4.13), that M has been chosen in (4.55) depending on n, N, ϵ, and we finally apply the Hölder inequality to have

$$\lambda_0^{2+\epsilon} \leq c_n \left[\fint_{Q_{2R}} \sum_r |Du_r|^{2+\epsilon} \, dz_r + \left(\fint_{Q_{2R}} \sum_r |F_r|^s \, dz_r \right)^{(2+\epsilon)/s} + 1 \right]^d$$

$$\leq c_n \left[\fint_{Q_{2R}} \sum_r |Du_r|^{2+\epsilon}\, dz_r + \left(\fint_{Q_{2R}} \sum_r |F_r|^{(2+\epsilon)(1+2\epsilon)}\, dz_r \right)^{1/(1+2\epsilon)} + 1 \right]^d, \tag{4.59}$$

where again $c_n \equiv c_n(n, N, \epsilon)$, and d is defined in (2.6). Joining the last estimate and (4.58), and reabsorbing at the left-hand side the last integral in (4.58) by a covering and iteration argument (see [13, Corollary 6.1, Lemma 6.1]), we obtain (2.5) for all $R \leq R_0$. At this stage, the constant c_n in (2.5) depends only on n, N, ϵ, and not on $\omega(\cdot)$; it is only R_0 which depends on $\omega(\cdot)$ through (4.56). The case $R \geq R_0$ follows from a standard covering argument; at this point c_n also depends on $\omega(\cdot)$ via the covering coefficients that depend on R_0. The proof of Theorem 2.1 is now complete. □

2.4 Proof of Theorem 2.2. By carefully looking at the proof of Theorem 2.1, we can immediately infer the statement of Theorem 2.2. Indeed, we first choose δ in (4.54) and M in (4.55); then, when dealing with (4.56), we may choose the number $\epsilon > 0$ in the statement of Theorem 2.2 small enough to have

$$\tilde{c}_n \epsilon^{(s-2-\epsilon)/s} \leq \frac{2\epsilon\delta^{1/(1+\epsilon)}}{6(1+2\epsilon)A^{1+2\epsilon}2^{(4+\epsilon)(1+2\epsilon)}}\epsilon_0,$$

and then we just estimate as follows:

$$\tilde{c}_n[\omega(R_0)]^{(s-2-\epsilon)/s} \leq \tilde{c}_n[a_r]_{BMO}^{(s-2-\epsilon)/s} \leq \frac{2\epsilon\delta^{1/(1+\epsilon)}}{6(1+2\epsilon)A^{1+2\epsilon}2^{(4+\epsilon)(1+2\epsilon)}}\epsilon_0. \tag{4.60}$$

The last inequality allows to recover (4.57), and the remainder of the proof works without any further change. This also implies that the constant $\tilde{c}_n$ in the final estimate (2.5) depends only on n, N, ϵ. □

Remark 4.1 The technique developed here allows us to get a rather precise dependence of the constant c_n in estimate (2.5) on the integrability parameter ϵ. We shall observe interesting similarities with the estimates obtained in the elliptic case (1.3) via maximal function techniques in [1], [10], and [14]; this is not surprising since the local use of estimates (3.8) and (3.11), in combination with Vitali's covering lemma, in some sense emulates the use of the maximal function. For the sake of simplicity, we confine ourselves to the model problem (1.1). We go back to (4.53); since $a_r(z_r) \equiv 1$, then we have $\omega(R) \equiv 0$. Then, taking δ and M in such a way as to obtain equalities in (4.54) and (4.55), respectively, and combining this with (4.53), recalling that B only depends on n, ϵ by (4.14), we obtain

$$\left(\fint_{Q_R} \sum_r |Du_r|^{(2+\epsilon)(1+2\epsilon)}\, dz_r \right)^{1/(1+2\epsilon)} \leq c_n A \lambda_0^{2+\epsilon} +$$

$$c_n \left(\frac{1+2\epsilon}{2\epsilon} A^{1+2\epsilon} 2^{(4+\epsilon)(1+2\epsilon)} \right)^{(2+\epsilon)/(1+\epsilon)} \left(\fint_{Q_R} \sum_r |F_r|^{(2+\epsilon)(1+2\epsilon)}\, dz_r \right)^{1/(1+2\epsilon)},$$

and therefore, by (4.59)

$$\left(\fint_{Q_R} \sum_r |Du_r|^{(2+\epsilon)(1+2\epsilon)}\, dz_r \right)^{1/(1+2\epsilon)} \leq c_n A \left[\fint_{Q_{2R}} \sum_r |Du_r|^{2+\epsilon}\, dz_r \right.$$

$$\left. + \left(\frac{1+2\epsilon}{2\epsilon} A^{1+2\epsilon} 2^{(4+\epsilon)(1+2\epsilon)} \right)^{(2+\epsilon)/(1+\epsilon)} \left(\fint_{Q_R} \sum_r |F_r|^{(2+\epsilon)(1+2\epsilon)}\, dz_r \right)^{1/(1+2\epsilon)} + 1 \right]^d. \tag{4.61}$$

The previous a priori estimate reveals the same asymptotic behavior for $1 + 2\epsilon \searrow 1$ both of the

constant appearing in the Hardy – Littlewood maximal function estimate (see [2]) and of the a priori elliptic estimates of [1], [10], and [14]. On the other hand, this is harmless: when $1 + 2\epsilon$ is approaching 1, a priori estimates bounding the $L^{(2+\epsilon)(1+2\epsilon)}$-norm of Du_r by the $L^{(2+\epsilon)(1+2\epsilon)}$-norm of F_r are simply given by Theorem 3.1, which works for any small choice of δ, with all constants remaining bounded as $\delta \searrow 0$, (see [16]). Note that the scaling of estimates (2.5) and (4.61) is in perfect accordance with that of the known a priori estimates for the evolutionary $(2 + \epsilon)$-Laplacian operator: when $F_r \equiv 0$, letting $1 + 2\epsilon \nearrow \infty$ in (4.61), we obtain, up to an absolute constant, the sup estimates (3.8) and (3.11) for $\theta = \gamma^2 = R^2$.

References

[1] E. Acerbi and G. Mingione, Gradient estimates for the $p(x)$-Laplacean system, J. Reine Angew. Math. 584 (2005), 117 – 148. MR 2155087 286, 313, 314

[2] B. Bojarski and T. Iwaniec, Analytical foundations of the theory of quasiconformal mappings in $\mathbb{R}^n$, Ann. Acad. Sci. Fenn. Ser. A I Math. 8 (1983), 257 – 324. MR 0731786 314

[3] M. Bramanti and M. C. Cerutti, $W_p^{1,2}$ solvability for the Cauchy-Dirichlet problem for parabolic equations with VMO coefficients, Comm. Partial Differential Equations 18 (1993), 1735 – 1763. MR 1239929 287

[4] S.-S. Byun, Parabolic equations with BMO coefficients in Lipschitz domains, J. Differential Equations 209 (2005), 229 – 265. MR 2110205 287

[5] L. A. Caffarelli and I. Peral, On $W^{1,p}$ estimates for elliptic equations in divergence form, Comm. Pure Appl. Math. 51 (1998), 1 – 21. MR 1486629 286, 287

[6] F. Chiarenza, M. Frasca,and P. Longo, $W^{2,p}$-solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients, Trans. Amer. Math. Soc. 336 (1993), 841 – 853. MR 1088476 287

[7] G. Cupini, N. Fusco, and R. Petti, Hölder continuity of local minimizers, J.Math. Anal. Appl. 235 (1999), 578 – 597. MR 1703712 295

[8] E. Dibenedetto, Degenerate Parabolic Equations, Universitext, Springer, New York, 1993. MR 1230384 286, 289, 291, 292, 295, 296, 299, 302, 304, 315, 317

[9] E. Dibenedetto and A. Friedman, H¨ older estimates for nonlinear degenerate parabolic systems. J. Reine Angew. Math. 357 (1985), 1 – 22. MR 0783531 291

[10] E. Dibenedetto and J. Manfredi, On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems, Amer. J. Math. 115 (1993), 1107 – 1134. MR 1246185 286, 313, 314

[11] F. Duzaar and A. Gastel, Nonlinear elliptic systems with Dini continuous coefficients, Arch. Math. (Basel) 78 (2002), 58 – 73. MR 1887317 318

[12] L. Escauriaza, Weak type-$(1, 1)$ inequalities and regularity properties of adjoint and normalized adjoint solutions to linear nondivergence form operators with VMO coefficients, Duke Math. J. 74 (1994), 177 – 201. MR 1271469 287

[13] E. Giusti, Direct Methods in the Calculus of Variations, World Sci., River Edge, N.J., 2003. MR 1962933 312

[14] T. Iwaniec, Projections onto gradient fields and L^p-estimates for degenerated elliptic operators, Studia Math. 75 (1983), 293 – 312. MR 0722254 286, 313, 314

[15] T. Iwaniec and C. Sbordone, Riesz transforms and elliptic PDEs with VMO coefficients, J. Anal. Math. 74 (1998), 183 – 212. MR 1631658 287

[16] J. Kinnunen and J. L. Lewis, Higher integrability for parabolic systems of p-Laplacian type, Duke Math. J. 102 (2000), 253 – 271. MR 1749438 286, 290, 314

[17] J. Kinnunen and J. L. Lewis, Very weak solutions of parabolic systems of p-Laplacian type, Ark.Mat. 40 (2002), 105 – 132. MR 1948889 286

[18] J. Kinnunen and P. Lindqvist, Summability of semicontinuous solutions to a quasilinear parabolic equation, Ann. Sc. Norm. Sup. Pisa Cl. Sci. (5) 4 (2005), 59 – 78. MR 2165403 286

[19] J. Kinnunen and S. Zhou, A local estimate for nonlinear equations with discontinuous coefficients, Comm. Partial Differential Equations, 24 (1999), 2043 – 2068. MR 1720770 286, 287, 316

[20] J. Kinnunen and S. Zhou, A boundary estimate for nonlinear equations with discontinuous coefficients, Differential Integral Equations 14 (2001), 475 – 492. MR 1799417 286

[21] J.-L. Lions, Quelques méthodes de résolution des problémes aux limites non linéaires, Dunod; Gauthier-Villars, Paris, 1969. MR 0259693 296

[22] M. Misawa, L^q estimates of gradients for evolutional p-Laplacian systems, J. Differential Equations 219 (2006), 390 – 420. MR 2183266 286, 289

[23] D. K. Palagachev, Quasilinear elliptic equations with VMO coefficients,Trans. Amer. Math. Soc. 347 (1995), 2481 – 2493. MR 1308019 287

[24] I. Peral and F. Soria, "A note on estimates for quasilinear parabolic equations" in Proceedings of the 2001 Luminy Conference on Quasilinear Elliptic and Parabolic Equations and System, Electron. J. Differ. Equ. Conf. 8, Southwest Texas State Univ., San Marcos, 2002, 121 – 131. MR 1990299 287

[25] D. Sarason, Functions of vanishing mean oscillation, Trans. Amer. Math. Soc. 207 (1975), 391 – 405. MR 0377518 288

[26] E. Acerbi and G. Mingione, Gradient estimates for a class of parabolic systems, Duke Math. J. 136 (2007), 285 – 320. DOI 10.1215/S0012-7094-07-13623-8