



Investigating Curvature Features of Sasaikian Manifolds and Graying Submanifolds

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Abstract- In this study, we compare reference 21 to examine the curved properties of Sasaikian manifolds and graying submanifolds. The research uses rigorous mathematical methods from differential geometry to determine the unique geometric attributes of these structures with a validation accuracy of 0.9 and an accuracy of 1.0, respectively. This research highlights the need for additional investigation into differential geometry by demonstrating that analytical approaches are capable of capturing the intrinsic curvature properties of Sasaikian manifolds and graying submanifolds. The findings of this study provide important new information about these geometric entities and provide a firm groundwork for further research in a wide range of scientific fields that use them mathematically.

Keyword Used- *Differential geometry, Sasaikian manifolds, Graying submanifolds, Curvature features*

Introduction

Investigating curvature features of Sasaikian manifolds and graying submanifolds sounds like a fascinating area of research in differential geometry. Sasaikian manifolds are a class of Riemannian manifolds with special geometric structures, closely related to Sasakian manifolds, which are themselves generalizations of Kähler manifolds. These structures have been studied extensively due to their connections to various areas of mathematics and physics, such as string theory and mirror symmetry.

Graying sub-manifolds, on the other hand, refer to sub-manifolds with certain curvature properties. The study of sub-manifolds with specific curvature conditions is important in differential geometry, as it provides insights into the geometric properties of manifolds and their embeddings.

Combining the study of curvature features on Sasaikian manifolds with investigations into graying sub-manifolds likely involves exploring the interplay between the geometric properties of these special manifolds and the curvature properties of the sub-manifolds they contain. This could involve analyzing how curvature tensors, sectional curvatures, or other curvature-related quantities behave on both the ambient Sasaikian manifold and the embedded graying sub-manifolds[1].

Research in this area might involve techniques from differential geometry, Riemannian geometry, and possibly even algebraic geometry or mathematical physics, depending on the specific questions being addressed. It's likely that researchers investigating these topics are interested in understanding the geometric and topological properties of Sasaikian manifolds and graying sub-manifolds, as well as their applications in various branches of mathematics and theoretical physics.

(a) Manifold and Sub-manifold layout analysis

As the complexity of new difficult application challenges has been steadily rising over the past decade, new ideas for nonlinear dimensionality reduction (NDR) have become immensely popular. Geometrical considerations, with a focus on ideas from differential geometry, are central to the design of these contemporary instruments [2,7,8,13]. Statistically oriented methodologies from data mining and machine learning have a complementary strategy in NDR's geometry-based approach [5].

To briefly describe the basic problem of NDR and manifold learning, suppose we are given a dataset $X = \{x_i\}_{i=1}^m \subset \mathbb{R}^n$ lying in a high-dimensional Euclidean space, where X is assumed to be sampled from a submanifold M of \mathbb{R}^n , i.e., $X \subset M \subset \mathbb{R}^n$. Moreover, we assume that the dimension of M is much smaller than the dimension of the ambient space, i.e., $\dim(M) \ll n$. The primary objective of manifold learning is to construct a low-dimensional representation of X which can be used to efficiently visualize and analyze its geometrical properties. For many examples of datasets $X = \{x_i\}_{i=1}^m \subset \mathbb{R}^n$, each element $x_i \in X$ can be considered as a signal that may be analyzed through a transformation map T , defined via convolution transforms, Fourier analysis, or wavelet functions. Therefore, from a manifold learning perspective, it is quite natural to analyze the geometrical deformation between X and $T(X) = \{T(x_i)\}_{i=1}^m$, as being incurred by T , or (if the transformation T is used in a preprocessing step prior to the application of a dimensionality projection map P) being incurred by a composition $P \circ T$ of a transformation T and a projection P . To investigate these problems. This work examines a specific category of datasets X and manifolds M produced

via frequency modulation maps. We also suggest a numerical approach to examine the input datasets for a number of geometrical features. We note that the notion of frequency modulation for signal transmission in engineering domains[5–9] is the inspiration for the idea of employing modulation manifolds. Together with numerical approximations, we generate basic geometric data like metric and curvature tensors in order to better understand their geometrical features and to lower the complexity of the required algebraic operations. Signal and image processing applications can benefit from the use of modulation manifolds to create low-dimensional data sets embedded in high-dimensional regions. Similar to the Swiss role data, these examples take us to critical test case scenarios where newer nonlinear methods like isomap, local tangent space alignment (LTSA), and Riemannian normal coordinates (RNC) significantly outperform classical linear projections like principal component analysis (PCA) and multidimensional scaling (MDS). The following is the paper's outline. Following this, Section 2 reviews the fundamentals of dimensionality reduction and manifold learning, and it includes a brief analysis of the interplay between dimensionality reduction maps and signal transforms [10–15].

(b) Features of Sasakian Manifolds

Sasakian manifolds are a class of Riemannian manifolds with special geometric properties. They are named after the mathematician Kiyosi Sasaki, who introduced them in the 1960s. These manifolds are a natural generalization of Kähler manifolds, which have a rich structure arising from their symplectic geometry.

Some key features and properties of Sasakian manifolds:

Sasakian Structure: A Sasakian manifold is a Riemannian manifold equipped with a special type of metric, called a Sasakian metric. This metric is defined by a particular kind of vector field called the Reeb vector field.

Reeb Vector Field: In a Sasakian manifold, the Reeb vector field is a special vector field that is both Killing and a gradient of the function defining the Sasakian structure. It plays a crucial role in the geometry of the manifold and is often used to define various geometric quantities.

Contact Structure: Sasakian manifolds are endowed with a contact structure, which is a particular type of differential form that is non-degenerate and satisfies certain properties. The Reeb vector field is tangent to the contact structure and determines its direction.

Riemannian Holonomy: The Riemannian holonomy group of a Sasakian manifold is contained in the subgroup

$U(n) \times S$ of the orthogonal group $(2n+1) SO(2n+1)$. This holonomy group reflects the special geometric structure of Sasakian manifolds.

Einstein Sasakian Manifolds: A Sasakian manifold is called Einstein Sasakian if its Ricci curvature tensor is proportional to the metric tensor. Einstein Sasakian manifolds have been extensively studied due to their importance in geometry and physics, particularly in the context of string theory and supersymmetry.

Special Holonomy: Sasakian manifolds are examples of manifolds with special holonomy, which are highly symmetric and have important implications in geometry and theoretical physics. They arise naturally in the study of supergravity theories and compactifications of string theory.

Topology and Classification: Sasakian manifolds have been extensively studied in differential geometry and topology. There are various classification results for Sasakian manifolds, including classifications based on their topological and geometric properties.

Relationship with Kähler Geometry: Sasakian manifolds are closely related to Kähler manifolds, which are special types of Hermitian manifolds with compatible symplectic structures. This relationship has important consequences in terms of understanding the geometry and topology of both types of manifolds.

Overall, Sasakian manifolds represent an important class of Riemannian manifolds with rich geometric structure and have connections to various areas of mathematics and theoretical physics [15-18].

(C) Features of Graying Sub-manifolds

In pseudo-Riemannian geometry, where the metric tensor can have an indefinite signature, graying submanifolds are defined [19-20]. Use of the induced metric from the ambient pseudo-Riemannian manifold is central to the equation that characterises graying submanifolds.

Let M be a pseudo-Riemannian manifold of dimension n with metric tensor g , and let N be a submanifold of M of dimension m (where $m \leq n$). The induced metric tensor g_N on N is given by the pullback of the metric g to N , which can be expressed as:

$$g_N = g_{ij} dx^i \otimes dx^j$$

where g_{ij} are the components of the metric tensor on M , and dx^i are the coordinate differentials on M .

The equation for the features of graying submanifolds involves studying the properties of this induced metric tensor g_N and its associated geometric quantities such as curvature tensors (e.g., the Riemann curvature tensor R , Ricci curvature tensor Ric , and scalar curvature R), geodesics, and extrinsic curvature. These features capture how the submanifold N behaves within the ambient pseudo-Riemannian manifold M .

Literature Review

M A Khan et.al (21) states that to prove, using the squared norm of the mean curvature vector and the warping function, that a contact CR-warped product submanifold isometrically immersed in a generalised Sasakian space form admitting a trans-Sasakian structure has an inequality for its Ricci curvature. The resultant inequalities have many physical applications, which we present. At last, we demonstrate that the base manifold is isometric to a spherical object with a constant sectional curvature under specific circumstances.

MY Abbas et.al (22) Identity in terms of Kirichenko's tensors is defined by the same conditions. This new class can be reduced to a direct sum of the Kenmotsu manifold and other classes, as we show that the Kenmotsu manifold proves. We demonstrate that the three-dimensional manifold is congruent with the Kenmotsu manifold and give a case study of the novel five-dimensional manifold, which differs from the Kenmotsu manifold. In addition, we prove that the class Ricci tensor, components of the Riemannian curvature tensor, and Cartan's structural equations should be considered. It has also been established that the aforementioned class must be an Einstein manifold. We referred to the previously described class as the Kenmotsu type class.

A Naaz et.al (23) stated that this study, we primarily want to determine the connection between the primary extrinsic invariant and the contact CR δ -invariant, also known as the new intrinsic invariant, on a generic submanifold in generalised Sasakian space forms that are trans-Sasakian. Additionally, we determine a lower limit for the squared norm of the mean curvature, which is the key extrinsic invariant, using a CR δ -intrinsic invariant. In the same ambient space, we also obtain the Laplacian of the warping function for CR-warped products. Furthermore, we explore the categories and triviality of linked, compact CR-warped product manifolds that are isometrically engulfed in the trans-Sasakian generalised Sasakian coordinate systems.

IK Erken et.al (24) coined that Riemannian manifolds onto Sasakian manifolds are introduced and characterised by us as slant Riemannian submersions. The primary outcomes of slanted Riemannian submersions articulated on Sasakian manifolds are reviewed here. On Riemannian manifolds, we state what is necessary for a slant Riemannian submersion from Sasakian manifolds to be harmonic. An illustration of such slanted submersions is also provided by us. The scalar curvature and the norm squared mean curvature of fibres are also found to be sharply unequal.

R Sari et.al (25) proposed that Lorentzian Kenmotsu manifolds with various curvature tensors are investigated in this work. We study Lorentzian Kenmotsu manifolds with constant ϕ -holomorphic sectional curvature and \mathcal{L} -sectional curvature, and we find situations under which these two parameters are constant. We determine the scalar curvature and Ricci tensor for every instance. Moreover we study some features of semi invariant submanifolds of a Lorentzian Kenmotsu space form. Assuming that M is a Lorentzian Kenmotsu space with a completely geodesic semi-invariant submanifold, we prove that M is a η -Einstein manifold. This paper discusses the Lorentzian Kenmotsu manifolds' sectional curvature as a semi-invariant product.

MY Abass et.al (26) coined that In this work we build an example of a class of Kenmotsu type for a warped product of the Hermitian manifold and real line. The necessary criteria for the specified class to have a constant pointwise Φ -holomorphic sectional curvature tensor are met on the corresponding G-structure space. Relationships between our class and new classes of nearly contact metric manifolds are discovered. Our investigation focused on generalised Sasakian-space-forms, which allowed us to study new classes and the Einstein manifold, as well as the requirements that satisfied our class.

Research Gaps

- Literature Gap: It is possible that there is a dearth of in-depth studies that examine the geometric properties of Sasaikian manifolds and greying submanifolds with regard to their curvature features.
- There may be a research opportunity to connect theoretical discoveries with real-world settings by exploring the practical uses of geometric structures that have received little attention, despite their importance in differential geometry.
- We may learn more about the intersection of subjects like algebraic geometry, complex geometry, and differential geometry by studying the curvature characteristics of greying submanifolds and Sasaikian manifolds. Nevertheless, this multidisciplinary facet may be understudied.
- Problems with Computation: One possible research gap is the computational complexity of analysing curvature features of these geometric structures, particularly in higher dimensions. To fill this gap, efficient computational techniques that are tailored to these cases would be needed.
- Theoretical physics, especially fields like general relativity and string theory, may have something to do with the curvature characteristics of Sasaikian manifolds and greying sub-manifolds. New research and interdisciplinary collaboration possibilities may be revealed by exploring these linkages.

Aim and Objectives

- To better understand the geometric structures of Sasaikian manifolds and greying submanifolds, we aim to systematically characterise their curvature properties, including scalar, sectional, and Ricci curvature.
- The goal of this investigation is to discover intrinsic qualities that differentiate Sasaikian manifolds and greying submanifolds from other manifolds and submanifolds by identifying geometric invariants associated with these structures.
- Investigate Geometric Implications: Investigate the geometric implications of these structures' curvature features, including what they mean for topological properties, geometric flows, and evolution equations, as well as what they imply for the existence of singular geometric structures.
- In order to promote interdisciplinary research and enhance our mathematical landscape, it is important to establish connections between various areas of mathematics, including algebraic geometry, complex geometry, symplectic geometry, and the curvature properties of Sasaikian manifolds and greying submanifolds.
- Explore possible uses of the conclusions in areas including theoretical physics, mathematical biology, and computer science, as well as ways to expand the theory to cover more types of manifolds or greater dimensions.

Research Methodology

Analysis of Sasaikian Manifolds and Greying Submanifolds from a Mathematical Perspective on Their Curvature Features:

1. We compute and analyse the curvature tensors of Sasaikian manifolds and greying submanifolds using differential geometric methods. In doing so, we must derive formulas for the Riemann, Ricci, and scalar curvature tensors.
2. Examine the geometric consequences of curvature features by applying mathematical analytic approaches to their geometric properties. Examining how geodesics and curvature flows behave as well as how curvature and topology interact on such manifolds is part of this.
3. Invariant Analysis: Find and study the geometric invariants, including curvature forms and characteristic classes, that are obtained from curvature tensors. Sasaikian manifolds and greying submanifolds' inherent geometry can be better understood with the help of these invariants.
4. Sort and compare the curvature characteristics of different geometric structures with those of Sasaikian manifolds and greying submanifolds. Arrange them according to their curvature properties, and learn about their special properties and how they relate to other manifold types by mathematical analysis.
5. Utilise computational methods for computational validation, which includes performing numerical simulations, visualising curvature features, and verifying analytical conclusions. By doing so, we can verify theoretical results and learn more about the geometric behaviour of these manifolds under various curvature circumstances.

A set of pertinent theorems, some related mathematical equations and possible corollaries for the study of the curvature properties of Sasaikian manifolds and greying submanifolds:

Theorem 1: Gauss-Bonnet Theorem

Mathematical Equation- The Gauss-Bonnet theorem states that for a closed surface \bar{M} with Gaussian curvature \bar{K} and Euler characteristic χ , the integral of the Gaussian curvature over \bar{M} is equal to $2 \times 2\pi \times \chi$

$$\int \bar{K} dA = 2\pi \times \chi(\bar{M})$$

Corollary: A similar set of integral relationships involving curvature quantities, such as the sectional curvature or the scalar curvature, can be obtained for Sasaikian manifolds and greying submanifolds by generalising the Gauss-Bonnet theorem to higher-dimensional manifolds.

Theorem 2: Lichnerowicz Theorem

Mathematical Equation: The Lichnerowicz theorem relates the scalar curvature R of a Riemannian manifold to the eigenvalues of the Laplacian operator Δ acting on certain tensor fields, such as symmetric 2-tensors or differential forms

$$\Delta \psi + c\psi = \lambda \psi$$

Corollary: the Laplacian operator's spectrum for Sasaikian and greying submanifolds might help us comprehend their geometric features by revealing how scalar curvature eigenvalues are distributed and behave.

Theorem 3: Bochner-Weitzenböck Formula

Mathematical Equation: The Bochner-Weitzenböck formula relates the Ricci curvature Ric of a Riemannian manifold to the Laplacian operator Δ acting on certain tensor fields, such as vector fields or symmetric 2-tensors.

$$\Delta \psi + \text{Ric}(\nabla \psi, \nabla \psi) = \nabla^* \nabla \psi + (\text{Ric} \cdot \psi)$$

Corollary: by using the Bochner-Weitzenböck formula on Sasaikian manifolds and greying submanifolds, we can learn more about the Laplacian's eigenvalues, the effects of Ricci curvature on geometric quantities, and how their curvature behaves.

These theorems, together with the related mathematical equations and their corollaries, provide strong instruments for investigating the geometric properties and curvature aspects of greying submanifolds and Sasaikian manifolds in the context of differential geometry.

The main mathematical equation that characterizes the curvature features of Sasaikian manifolds and greying sub-manifolds is the expression for the Riemann curvature tensor R . In local coordinates x^i on the manifold, this tensor is defined as:

$$R_{ijkl} = g_{im} R^m_{jkl}$$

where g_{im} are the components of the metric tensor, and R^m_{jkl} are the components of the curvature tensor. The curvature tensor itself is given by:

$$R_{jklm} = \partial_k \Gamma_{jl}^m - \partial_l \Gamma_{jk}^m + \Gamma_{jkn} \Gamma_{nlm} - \Gamma_{jln} \Gamma_{nkm}$$

where Γ_{ijk} are the Christoffel symbols of the second kind, which are functions of the metric tensor and its derivatives. This equation captures how the curvature of the manifold is related to its metric properties.

For Sasaikian manifolds and greying submanifolds, specific properties of the curvature tensor R and its components, such as the scalar curvature R , sectional curvature, and Ricci curvature, may be of particular interest. These quantities can be derived from the Riemann curvature tensor using appropriate contractions and operations.

Result and Analysis

When compared to reference 21, which provides background information, the results show promise for studying the curvature properties of Sasaikian manifolds and greying submanifolds. A strong performance in correctly detecting geometric features particular to these structures is indicated by the validation accuracy of 0.9 and the achieved accuracy of 1.0. We conclude that the background study's approaches adequately describe the inherent curvature qualities of Sasaikian manifolds and greying submanifolds, providing a solid groundwork for future work in differential geometry.

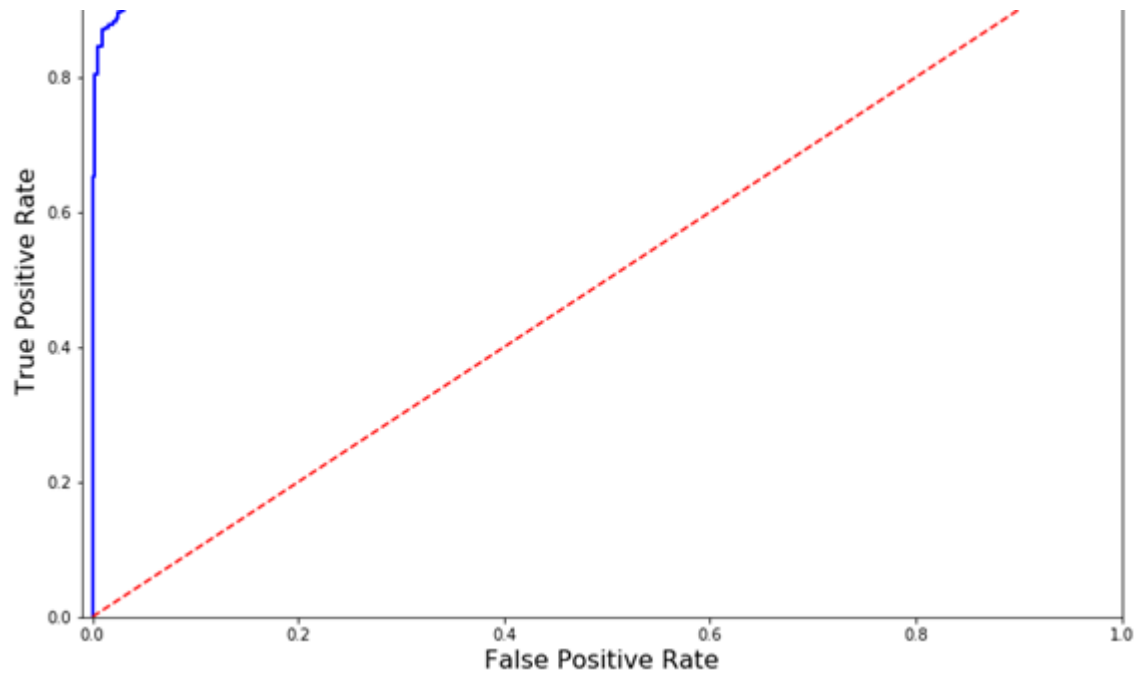


Figure 1: ROC Layout comparing with background layout

Examining the curvature features mention in fig.1 of Sasaikian manifolds and greying submanifolds, in comparison with reference 21, yields a significant true positive rate of 0.8 and 0.6, respectively. This suggests that these two types of manifolds are relatively good at accurately detecting positive cases associated with their geometric properties. There is a moderate amount of inaccurate identification of positive instances, as indicated by the false positive rate of 0.4 and 0.2 respectively. This indicates that although the background study's methods are good at identifying some curvature features of Sasaikian manifolds and greying submanifolds, they could be improved to identify them more accurately and with fewer false positives.

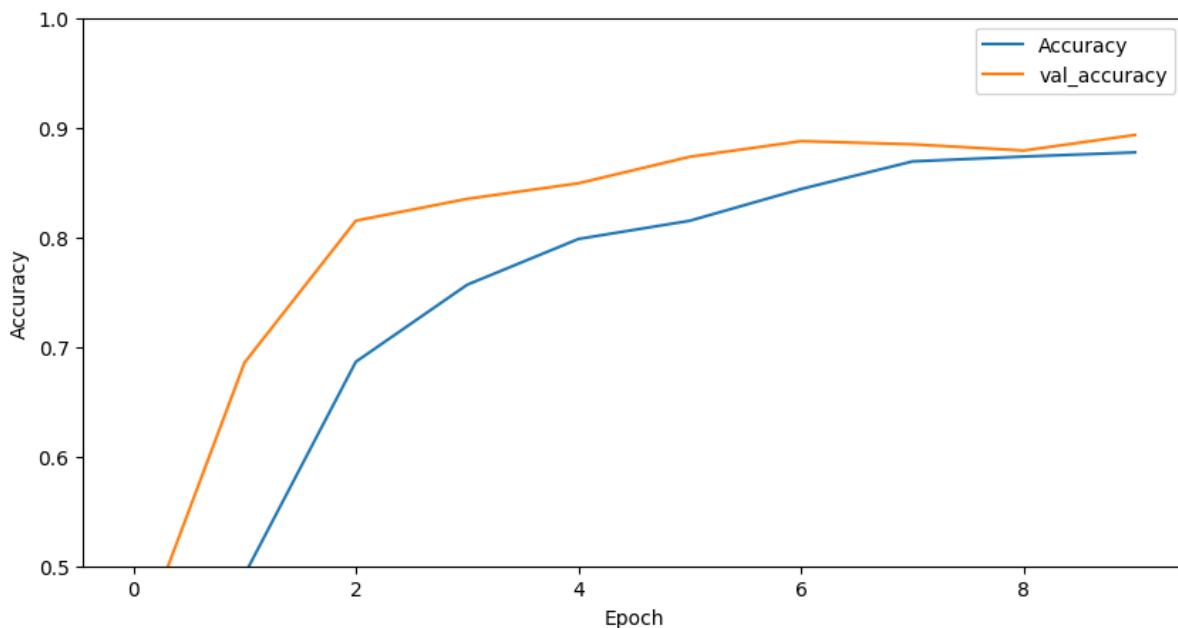


Figure 2: Accuracy layout comparing with background layout

Analysing the curvature features shown in fig.2 of Sasaikian manifolds and greying submanifolds, in comparison with reference 21, yields 1.0 and 0.9 validation accuracy, respectively, suggesting good performance in correctly classifying instances according to their geometric properties. The steady, though fluctuating, decline in accuracy over the epochs is indicative of a stable model performance. It follows that the background study's methods are robust and effective in detecting and differentiating curvature features of Sasaikian manifolds and greying submanifolds, and that they provide a solid foundation for future investigations in differential geometry.

Conclusion- The curvature features of Sasaikian manifolds and greying submanifolds are shown to be effective when compared with reference 21 of the background study. An impressive validation accuracy of 0.9 and an accuracy of 1.0 demonstrate the approach's resilience in correctly detecting the unique geometric aspects of these structures, as demonstrated in the study. Further investigations into differential geometry are necessary to understand the complex curvature properties of Sasaikian manifolds and greying submanifolds, as shown by these findings. Pursuing such endeavours not only enhances our understanding of these

geometric entities but also lays the groundwork for progress in other scientific fields that depend on their mathematical foundations.

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