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b_2 - Metric Space and Some Fixed Point Results

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Abstract : In this paper, we have used the concept of b_2 - metric space. We aim to obtain the common coincidence and common fixed-point theorems for two mappings on in b_2 - metric spaces. The concept of 2 - metric space was introduced by Gahler. Many fixed-point results were also obtained for mappings defined on these spaces. Later Mustafa gave the new concept of b_2 - metric spaces which are the generalization of both b-metric space and 2 - metric space. Some Fixed-point results for the mappings satisfying the contractive type conditions are obtained on the b_2 - metric spaces.

Keywords: **Fixed Point, b_2 - Metric Space, Convergence in b_2 – Metric Space, Cauchy Sequence.**

I. Introduction

The Banach fixed point theorem is very popular and useful theorem in Mathematics as well as in other subjects. In 1989, Bakhtin [1] introduced the concept of generalized b–metric spaces. Boriceanu [2], and Mehmat Kir [3] extended the fixed point theorem in b–metric space. Borkar [4] obtained the common fixed point theorem for non-expansive type mapping. Czerwik [5-6] presented the generalization of the Banach fixed point theorem in b-metric spaces. Using this idea, many researchers presented a generalization of the renowned Banach fixed point theorem in b-metric space. Agrawal [7] presented the existence and uniqueness theorem in b–metric Space. Chopade [8] gave common fixed point theorems for contractive type mapping in metric space. Borgaonkar V. D. and K. L. Bondar [9-10] have obtained the fixed point theorems in b -metric spaces. Roshan [11] obtained a common fixed point of four maps in b–Metric space. Suzuki [12] obtained some basic inequalities and it's applications in a b – metric space. In this paper, we will obtain the fixed point theorem for a pair of mappings in b - metric space.

The notion of a 2-metric was introduced by Gähler in [14], having the area of a triangle in R^2 as the inspirative example. Similarly, several fixed-point results were obtained for mappings in such spaces. Note that, unlike many other generalizations of metric spaces introduced recently, 2-metric spaces are not topologically equivalent to metric spaces and there is no easy relationship between the results obtained in 2-metric and in metric spaces. A generalization of both 2-metric space and b-metric space is introduced as a b_2-metric space by Zead Mustafa [15] in 2014.

In this paper, we prove some fixed-point theorems under various contractive conditions in b_2-metric spaces.

II. SOME BASIC DEFINITIONS AND PRELIMINARIES

Definition 1.1: Let X be a nonempty set and let $d: X^3 \to R$ be a map satisfying the following conditions:

1. For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.
2. If at least two of the three points x, y, z are same then $d(x,y,z) = 0$.
3. Symmetry:$d(x,y,z) = d(x,z,y) = d(y,z,x) = d(y,x,z) = d(z,y,x) = d(z,x,y)$.
4. Rectangular Inequality: $d(x,y,z) \leq d(x,y,a) + d(y,z,a) + d(x,z,a)$

Then d is called as a 2-Metric on X and (X,d) is called as a 2- Metric space.

Example 1.1: Let $X \times X \times X \to R$ be defined by,

$$d(x,y,z) = min\{|x-y|, |y-z|, |z-x|\}$$

Clearly d is a 2-metric on X.

Definition 1.2: Let X be a non-empty set and $s \geq 1$ then $d: X^3 \to R$ satisfying following conditions:

1. For every pair of distinct point $x, y \in X$, there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.
2. If at least two of the three points x, y, z are same then $d(x,y,z) = 0$.
3. Symmetry:$d(x,y,z) = d(x,z,y) = d(y,z,x) = d(y,x,z) = d(z,y,x) = d(z,x,y)$.
4. Rectangular Inequality: $d(x,y,z) \leq s\{d(x,y,a) + d(y,z,a) + d(x,z,a)\}$

Then d is called as a b_2-metric on X and (X,d,s) is called as a b_2-metric space.

Definition 1.3: Let $\{x_n\}$ be a sequence in a b_2-metric space (X,d). Then

1. $\{x_n\}$ is said to be b_2-convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, if for all $a \in X$, $\lim_{n\to\infty} d(x_n, x, a) = 0$.
2. $\{x_n\}$ is said to be b_2- Cauchy sequence in X if for all $\lim_{n\to\infty} d(x_n, x_m, a) = 0$.
3. (X, d) us said to be b_2-complete if every b_2- Cauchy sequence is a b_2-convergent.

Example 1.2: Let $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{n}, \ldots\right\}$ and the mapping $d_1 = X \times X \times X \to R$ be defined by,

$$d_1(x, y, z) = \begin{cases} (xy + yz + zx)^2 & if\ x \neq y \neq z \\ 0 & otherwise \end{cases}$$

clearly, d_1 is a b_2-metric on X and (X, d_1, s) is a b_2-metric space.

Example 1.3: Let $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{n}, \ldots\right\}$ and the mapping $d_1 = X \times X \times X \to R$ be defined by, $d_2(x, y, z) = (x - y)^2(y - z)^2(z - x)^2$ clearly, d_2 is a b_2-metric on X. (X, d_2, s) is a b_2-metric space.

III. MAIN RESULT

Theorem 3.1: Let (X, d, s) be a Complete b_2-metric space with $s \geq 1$ $P, Q: X \to X$ be a self-maps on X, such that,

$d(Px, Qy, a) \leq \alpha[d(Px, y, a) + d(x, Qy, a)] + \beta[d(Px, x, a) + d(Qy, y, a)]$ ... ...(3.1)

holds for $\forall\, x, y, a \in X$. Where, $0 < \alpha < \frac{1}{s}$ and $0 < \beta < \frac{1}{s}$, such that, $\alpha + \beta < \frac{1}{2s}$.

Then, P and Q have a unique common fixed point in X.

Proof:

Let $x_0 \in X$ be an arbitrary element of X. We define a sequence $\{x_n\}$ of distinct points in X as,

$x_{2n+1} = Px_{2n}$ and $x_{2n+2} = Qx_{2n+1}$ $\forall\, n = 0, 1, 2$ ... ... (3.2)

we claim that $\{x_n\}$ is Cauchy's sequence.

Now, we will prove that, $d(x_n, x_{n+1}, x_{n+2}) = 0 \quad \forall\, n \in N$.

Firstly, we will prove that $d(x_{2n}, x_{2n+1}, x_{2n+2}) = 0$

Suppose $d(x_{2n}, x_{2n+1}, x_{2n+2}) > 0 \ \ \forall\, n \in N$

$\therefore\ d(x_{2n}, x_{2n+1}, x_{2n+2}) = d(x_{2n+2}, x_{2n+1}, x_{2n})$

$d(Qx_{2n+1}, Px_{2n}, x_{2n}) = d(Px_{2n}, Qx_{2n+1}, x_{2n})$

$\leq \alpha[d(x_{2n+1}, x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n+2}, x_{2n})] + \beta[d(x_{2n+1}, x_{2n}, x_{2n}) + d(x_{2n+2}, x_{2n+1}, x_{2n})]$

$\therefore\ d(x_{2n}, x_{2n+1}, x_{2n+2}) \leq \beta . d(x_{2n+2}, x_{2n+1}, x_{2n})$

$\therefore\ d(x_{2n}, x_{2n+1}, x_{2n+2}) < d(x_{2n+2}, x_{2n+1}, x_{2n})$

This is not possible.

$\therefore$ we have $d(x_{2n}, x_{2n+1}, x_{2n+2}) = 0 \quad \forall\, n \in N$ ... ... (3.3)

Now, we will show that $d(x_{2n+1}, x_{2n+2}, x_{2n+3}) = 0 \quad \forall\, n \in N$.

Suppose, $d(x_{2n+1}, x_{2n+2}, x_{2n+3}) > 0$ i. e. $d(x_{2n+1}, x_{2n+2}, x_{2n+3}) > 0$.

Consider,

$d(x_{2n+1}, x_{2n+2}, x_{2n+3}) = d(x_{2n+3}, x_{2n+2}, x_{2n+1}) = d(Px_{2n+2}, Qx_{2n+1}, x_{2n+1})$

$\leq \alpha[d(Px_{2n+2}, x_{2n+1}, x_{2n+1}) + d(x_{2n+2}, Qx_{2n+1}, x_{2n+1})] + \beta\big[d(Px_{2n+2}, x_{2n+2}, x_{2n+1}) + d\big(Qx_{2n+1}, x_{2n+1,}\, x_{2n+1}\big)\big]$

$\leq \alpha[d(Px_{2n+2}, x_{2n+1}, x_{2n+1}) + d(x_{2n+2}, x_{2n+2}, x_{2n+1})] + \beta\big[d(x_{2n+3}, x_{2n+2}, x_{2n+1}) + d\big(x_{2n+2}, x_{2n+1,}\, x_{2n+1}\big)\big]$

$d(x_{2n+1}, x_{2n+2}, x_{2n+3}) \leq \beta d(x_{2n+1}, x_{2n+2}, x_{2n+3}) < d(x_{2n+1}, x_{2n+2}, x_{2n+3})$

This is not possible.

$\therefore$ we have, $d(x_{2n+1}, x_{2n+2}, x_{2n+3}) = 0 \quad \forall\, n \in N$ ... ... (3.4)

In general, we have, $d(x_n, x_{n+1}, x_{n+2}) = 0 \quad \forall\, n \in N$. ... ... (3.5)

Now we will prove that, $\{x_n\}$ is Cauchy's sequence in X.

Consider, for $x_{2n+1} \neq a$ and $x_{2n} \neq a$,

$d(x_{2n+1}, x_{2n}, a) = d(Px_{2n}, Qx_{2n-1}, a)$

$\leq \alpha[d(Px_{2n}, x_{2n-1}, a) + d(x_{2n}, Qx_{2n-1}, a)] + \beta[d(Px_{2n}, x_{2n}, a) + d(Qx_{2n-1}, x_{2n-1}, a)]$

$\leq \alpha[d(x_{2n+1}, x_{2n-1}, a) + d(x_{2n}, x_{2n}, a)] + \beta[d(x_{2n+1}, x_{2n}, a) + d(x_{2n}, x_{2n-1}, a)]$

$\leq \alpha[sd(x_{2n+1}, x_{2n}, a) + sd(x_{2n}, x_{2n-1}, a) + sd(x_{2n}, x_{2n-1}, x_{2n+1})] + \beta[d(x_{2n+1}, x_{2n}, a) + d(x_{2n}, x_{2n-1}, a)]$

Now, (3.5) gives,

$d(x_{2n+1}, x_{2n}, a) \leq \alpha[sd(x_{2n+1}, x_{2n}, a) + sd(x_{2n}, x_{2n-1}, a)] + \beta[d(x_{2n+1}, x_{2n}, a) + d(x_{2n}, x_{2n-1}, a)]$

$\leq (\alpha s + \beta)\, d(x_{2n+1}, x_{2n}, a) + (\alpha s + \beta)\, d(x_{2n}, x_{2n-1}, a)(1 - \alpha s - \beta)\, d(x_{2n+1}, x_{2n}, a)$

$\leq\ (\alpha s + \beta)\, d(x_{2n}, x_{2n-1}, a)$

$d(x_{2n+1}, x_{2n}, a) \leq \left[\frac{\alpha s+\beta}{1-\alpha s-\beta}\right] d(x_{2n}, x_{2n-1}, a)$
$d(x_{2n+1}, x_{2n}, a) \leq r. d(x_{2n}, x_{2n-1}, a)$.

Where, $r = \left[\frac{\alpha s+\beta}{1-\alpha s-\beta}\right] < 1$.

By continuing we get, $\forall\, a \neq x_{n+2},\ a \neq x_{n+1}$
$d(x_{2n+1}, x_{2n}, a) < r. d(x_{2n}, x_{2n-1}, a) < r^2. d(x_{2n-1}, x_{2n-2}, a) < \cdots < r^{2n}. d(x_1, x_0, a)$.
In general, we have, $d(x_{n+2}, x_{n+1}, a) < r^{n+1}. d(x_1, x_0, a)$.
Hence, we have,

$$\lim_{n\to\infty} d(x_{n+2}, x_{n+1}, a) = \lim_{n\to\infty} d(x_n, x_{n+1}, a) = 0 \qquad \ldots\ldots\ldots(3.6)$$

Therefore, by definition, for every $\epsilon > 0$, there exists a positive integer N_1 such that,

$$d(x_{n+1}, x_n, a) < \frac{\epsilon}{3s} \quad \forall\ n \in N_1 \qquad \ldots\ldots\ldots(3.7)$$

Now we will prove that $\{x_n\}$ is Cauchy's sequence in X. i.e., for every $\epsilon > 0$, there exists a positive number N_0 such that,

$$d(x_m, x_n, a) < \epsilon \quad \forall\ m > n \geq N_0 \qquad \ldots\ldots\ldots(3.8)$$

We will prove (3.8) by using the method of induction.
Case-I If $m = n + 1$, then by (3.7), we get,

$$d(x_m, x_n, a) = d(x_{n+1}, x_n, a) < \frac{\epsilon}{3s} \quad \forall\ n \in N_1$$

Hence, equation (3.8) holds for $m = n + 1$
Case-II Now we will prove that (3.8) holds for $m = m'$. Then we have,

$$d(x_n, x_{m'}, a) < \frac{\epsilon}{3s} \quad \forall\, m > n \geq N_2 \qquad \ldots\ldots\ldots(3.9)$$

Case-III Now we will prove that (3.8) holds for $m = m' + 1$.
Consider,

$$d(x_n, x_{m'+1}, a) \leq sd(x_n, x_{m'}, a) + sd(x_{m'}, x_{m'+1}, a) + sd(x_n, x_{m'}, x_{m'+1})$$

By (3.7) and (3.9) we have,

$$d(x_n, x_{m'+1}, a) < s.\left[\frac{\epsilon}{3s} + \frac{\epsilon}{3s} + \frac{\epsilon}{3s}\right] \quad \forall\, m' > n \geq N_2$$

Thus, by the principle of induction, (3.8) holds for all $m > n$.
Therefore $\{x_n\}$ is Cauchy's sequence in X.
As X is complete $\{x_n\}$ is a convergent sequence in X.
Suppose, $\lim_{n\to\infty} x_n = x^*,\ x^* \in X$. Hence, $\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} x_{2n+1} = x^* \qquad \ldots\ldots\ldots(3.10)$

Now we will prove that x^* is fixed point of both P and Q i.e. we claim that, $Px^* = Qx^* = x^*$.

If possible, suppose that, $Px^* \neq x^*$.
Thus, $\forall\, a \neq Px^*$ *and* $a \neq x^*$ we have $d(Px^*, x^*, a) \neq 0$.
$\therefore d(Px^*, x^*, a) > 0$.

Consider $\forall\, a \neq Px^*$ *and* $a \neq x^*$,
$d(Px^*, x^*, a)$
$\leq sd(Px^*, x_{2n}, a) + sd(x_{2n}, x^*, a) + sd(Px^*, x^*, x_{2n})$
$\leq sd(Px^*, Qx_{2n-1}, a) + sd(x_{2n}, x^*, a) + sd(Px^*, x^*, x_{2n})$

$\leq s\{\alpha[d(Px^*, x_{2n-1}, a) + d(x^*, Qx_{2n-1}, a)] + \beta[d(Px^*, x^*, a) + d(Qx_{2n-1}, x_{2n-1}, a)]\} + sd(x_{2n}, x^*, a) + sd(Px^*, x^*, x_{2n})$

$\leq s\{\alpha[d(Px^*, x_{2n-1}, a) + d(x^*, Qx_{2n-1}, a)] + \beta[d(Px^*, x^*, a) + d(x_{2n}, x_{2n-1}, a)]\} + sd(x_{2n}, x^*, a) + sd(Px^*, x^*, x_{2n})$

$\leq s\{\alpha[d(Px^*, x_{2n-1}, a) + d(x^*, x_{2n}, a)] + \beta[d(Px^*, x^*, a) + d(x_{2n}, x_{2n-1}, a)]\} + sd(x_{2n}, x^*, a) + sd(Px^*, x^*, x_{2n})$

Letting $\lim_{n\to\infty}$ on both sides then we get,

$$\lim_{n\to\infty} d(Px^*, x^*, a) \leq \lim_{n\to\infty} s.\{\alpha[d(Px^*, x_{2n-1}, a) + d(x^*, x_{2n}, a)] + \beta[d(Px^*, x^*, a) + d(x_{2n}, x_{2n-1}, a)]\} + \lim_{n\to\infty} s. d(x_{2n}, x^*, a) + \lim_{n\to\infty} s. d(Px^*, x^*, x_{2n})$$

Thus, (3.6) and (3.10) gives,
$d(Px^*,x^*,a) \leq s.\{\alpha[d(Px^*,x^*,a)+d(x^*,x^*,a)]+\beta[d(Px^*,x^*,a)]\} + s.d(x^*,x^*,a)+s.d(Px^*,x^*,x^*)$

$d(Px^*,x^*,a) \leq s.\{\alpha[d(Px^*,x^*,a)]+\beta[d(Px^*,x^*,a)]\}$
$d(Px^*,x^*,a) \leq (\alpha+\beta)s.d(Px^*,x^*,a)$
$d(Px^*,x^*,a) < \frac{1}{2}d(Px^*,x^*,a)$
This is a contradiction since $d(Px^*,x^*,a) > 0$.
Therefore, we have,
$Px^* = x^*$
Therefore, x^* is fixed point of P.

Now. we show that, x^* is fixed point of Q i. e. $d(Qx^*,x^*,a) = 0$.
Suppose, $\forall\, a \neq Qx^*$ and $a \neq x^*$ $d(Qx^*,x^*,a) > 0$.

Consider $\forall\, a \neq Qx^*$ and $a \neq x^*$,
$d(Qx^*,x^*,a)$
$\leq sd(Qx^*,x_{2n+1},a)+sd(x_{2n+1},x^*,a)+sd(Qx^*,x^*,x_{2n+1})$
$\leq sd(Qx^*,Px_{2n},a)+sd(x_{2n+1},x^*,a)+sd(Qx^*,x^*,x_{2n+1})$
$\leq sd(Px_{2n},Qx^*,a)+sd(x_{2n+1},x^*,a)+sd(Qx^*,x^*,x_{2n+1})$
$\leq s\big[\alpha[d(Px_{2n},x^*,a)+d(x_{2n},Qx^*,a)]+\beta[d(Px_{2n},x_{2n},a)+d(Qx^*,x^*,a)]\big] + sd(x_{2n+1},x^*,a)+sd(Qx^*,x^*,x_{2n+1})$

$\leq s\big[\alpha[d(x_{2n+1},x^*,a)+d(x_{2n},Qx^*,a)]+\beta[d(x_{2n+1},x_{2n},a)+d(Qx^*,x^*,a)]\big] + sd(x_{2n+1},x^*,a)+sd(Qx^*,x^*,x_{2n+1})$

Letting $\lim_{n\to\infty}$ on both sides then we get,
$$\lim_{n\to\infty} d(Qx^*,x^*,a) \leq \lim_{n\to\infty} s\big[\alpha[d(x_{2n+1},x^*,a)+d(x_{2n},Qx^*,a)]+\beta[d(x_{2n+1},x_{2n},a)+d(Qx^*,x^*,a)]\big] + \lim_{n\to\infty} sd(x_{2n+1},x^*,a)+\lim_{n\to\infty} sd(Qx^*,x^*,x_{2n+1})$$

Thus, (3.6) and (3.10) gives,
$d(Qx^*,x^*,a)$
$\leq s[\alpha[d(x^*,x^*,a)+d(x^*,Qx^*,a)]+\beta.d(Qx^*,x^*,a)]+sd(x^*,x^*,a)+sd(Qx^*,x^*,x^*)$
$\leq s[\alpha[d(x^*,Qx^*,a)]+\beta.d(Qx^*,x^*,a)]$
$d(Qx^*,x^*,a) \leq (\alpha+\beta)s.d(Qx^*,x^*,a)$
$d(Qx^*,x^*,a) < \frac{1}{2}d(Qx^*,x^*,a)$
This is a contradiction since $d(Qx^*,x^*,a) > 0$.
Therefore, we have,
$Qx^* = x^*$.
Therefore, x^* is fixed point of Q.
Hence, we have, $Px^* = Qx^* = x^*$ x^* is common fixed point of P and Q.
Now we will prove that x^* is a unique common fixed point of P and Q.
Suppose y^* is another common fixed point of P and Q.
$\therefore Py^* = Qy^* = y^*$
Now, we will prove that, $x^* = y^*$.
Suppose $x^* \neq y^*$
Therefore, $\forall\, a \neq x^*$ and $a \neq y^*$ we have $d(x^*,y^*,a) \neq 0$
Suppose $d(x^*,y^*,a) > 0$.

Consider, $\forall\, a \neq x^*$ and $a \neq y^*$
$d(x^*,y^*,a) = d(Px^*,Qy^*,a)$
$\leq \alpha[d(Px^*,y^*,a)+d(x^*,Qy^*,a)]+\beta[d(Px^*,x^*,a)+d(Qy^*,y^*,a)]$
$\leq \alpha[d(x^*,y^*,a)+d(x^*,y^*,a)]+\beta[d(x^*,x^*,a)+d(y^*,y^*,a)]$
$\leq \alpha[d(x^*,y^*,a)+d(x^*,y^*,a)]$
$\leq 2\alpha\, d(x^*,y^*,a)$
This is not possible,

Therefore, we have $x^* = y^*$
Hence, x^* unique fixed point for the mapping P.
This completes the proof.

Corollary 3.1: Let (X, d, s) be a Complete b_2 −metric space with $s \geq 1$. $P, Q: X \to X$ are self-maps on X. Let any one of P and Q is continuous. Suppose, P and Q satisfies (3.1), then P and Q has a unique common fixed point in X.
Corollary 3.2: Let (X, d, s) be a Complete b_2 −metric space with $s \geq 1$. $P, Q: X \to X$ are both continuous self-maps on X satisfying (3.1), then P and Q has a unique common fixed point in X.

IV. DISCUSSION AND THE CONCLUDING REMARK

In this paper, the $b_2 -$ metric under consideration is not necessarily continuous we have proved the existence and uniqueness of common fixed points for two mappings satisfying contractive type conditions in a $b_2 -$ metric space.

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