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Exact and Numerical solution of Fuzzy ordinary differential equations

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Abstract : Fuzzy differential equation is extremely useful to the differential equations where the variables are considered to be in a range. In this paper we introduce exact and numerical solution of first order simultaneous fuzzy differential equation. Fifth order Runge-Kutta Merson Method based on Seikkala derivative of fuzzy differentiation is used to solve numerically. An example is given to compare exact and numerical solution.

Index Terms **– Fuzzy differential equation, Fuzzy sets, exact solutions, Runge Kutta Methods**

1. Introduction

A common approach in modelling real world problems is ordinary differential equations. This approach requires precise information about the problem under investigation. In many cases, this information will often not be known precisely. This is obvious when determining initial values, parameter values, or functional relationships. In order to handle such situations, the use of fuzzy sets may be seen as an effective tool for a better understanding of the studied phenomena. It is therefore not surprising that there is a vast literature dealing with fuzzy differential equations.

In the literature, there are several approaches to study fuzzy differential equations. The first and the most popular one is Hukuhara derivative. However, this approach suffers from certain disadvantages since the solution of a fuzzy differential equation becomes fuzzier and fuzzier as independent variable increases. According to Diamond, this approach does not reproduce the rich and varied behaviour of ordinary differential equations. Therefore, it is not suitable for modelling. In order to overcome this problem, H¨ullermeier has interpreted a fuzzy differential equation as a set of differential inclusions, i.e expressing the solution of a fuzzy differential equation as level setwise. However, in general, the solution is not convex on the time domain.

Seikkala differentiability is simpler and stronger than the other differentiability. Unfortunately, there are many elementary fuzzy-valued functions which frequently occur in solution of fuzzy differential equations, are not Seikkala differentiable. The purpose of the paper is to generalize the Seikkala differentiability. We see that a larger class of elementary fuzzy-valued functions belong to a generalized Seikkala differentiability (gS-differentiability) concept. We find the fuzzy solution of a fuzzy initial value problem using gSdifferentiability of a fuzzy-valued function.

All the ideas are based on extending the existing classical numerical methods to the fuzzy case. This process yields fuzzy Euler method [5, 7, 18, 21, 23], fuzzy Taylor method [1], fuzzy Runge-Kutta methods and many more. Efficient computational algorithms developed in [6, 8] can be incorporated in order to guarantee the convexity of fuzzy solution on the time domain. In this paper, we study the relationship between the several varieties of fuzzy Euler method for solving fuzzy differential equations.

The paper is organized in following manner. Section 2 contains basics of fuzzy numbers. The generalized Seikkala differentiability of a fuzzy-valued function is proposed and compared with Seikkala differentiability with appropriate examples in Section 3. The properties of gS differentiability are discussed in the same section. Runge Kutta methods of 5^{th} order is discussed. A numerical wxample is given to illustrate the result. Conclusion is given in the following section.

2 Fuzzy numbers

Fuzzy numbers have been introduced by Lotfi A. Zadeh to handle nonspecific numerical quantities in a practical way. For instance, profit on some product is approximately known, 2 Rs, say, then we can express this approximate amount 2 by means of fuzzy number. The numbers which are near to 2, can also included in fuzzy number 2 with varying membership grade. That is, 1 is regarded as approximately 2 with degree of membership 0.5. Similarly, 3 can be regarded as approximately 2 with degree of membership 0.6. In general, when numbers are imprecise/inexact/partially known, they can be represented as fuzzy numbers rather than real numbers. The mathematical definition of fuzzy number is given as follows.

- **Fuzzy Set**
  - Let X be a collection elements (universe of discourse) and x is an element of X, then a fuzzy set $\tilde{A}$ in X is defined by the membership function $\mu_{\tilde{A}}(x)$, $0 \le \mu_{\tilde{A}}(x) \le 1$, defined on every element of $\tilde{A}$. A fuzzy set $\tilde{A}$ will be denoted by $\tilde{A}=\left\{\left(x,\mu_{\tilde{A}}(x)\right): x \in X\right\}$ or $\tilde{A}=\left\{\sum \mu_{\tilde{A}}(x)/x, x \in X\right\}$ or $\tilde{A}= \int_{x\in X} \mu_{\tilde{A}}(x)/x$.

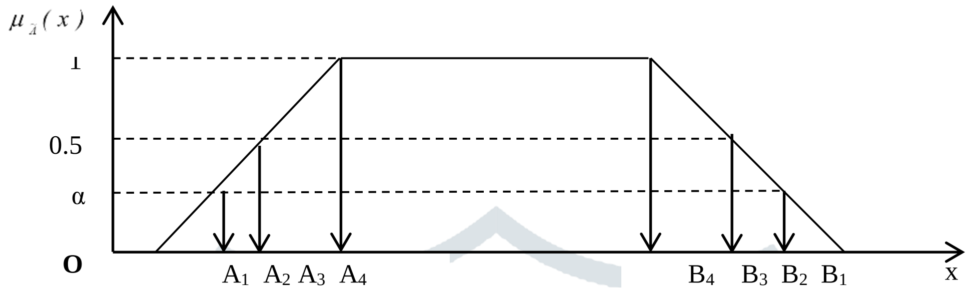


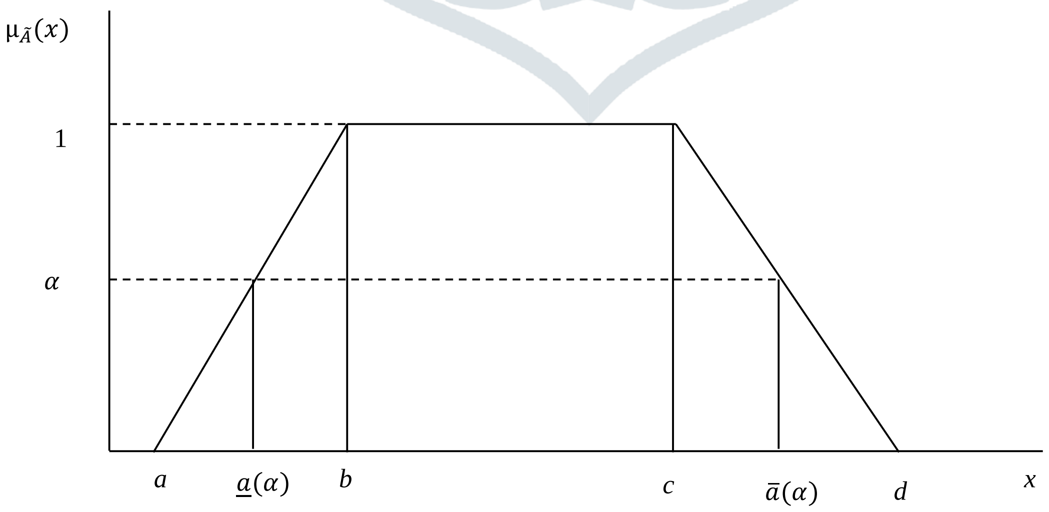
Figure 1 : Support (A_1B_1), α-cut (A_2B_2), Crossover (A_3B_3), Core (A_4B_4) of a FS $\tilde{A}$

- **Fuzzy Number**

A fuzzy number is defined as $\tilde{A}=\left\{\left(x,\mu_{\tilde{A}}(x)\right), x \in \Re\right\}, \Re : -\infty < x < \infty,$ and its membership function $\mu_{\tilde{A}}(x)$ is a mapping $\mu_{\tilde{A}} : \Re \rightarrow [0,1]$ where $'x'$ represents the alternatives.

- **Trapezoidal Fuzzy Number**

  A fuzzy number $\tilde{M} = (a, b, c, d)$ is called trapezoidal fuzzy number (TrFN), shown in adjacent figure1,



where $a \le b \le c \le d$, if its membership functions has the following form (Tiryaki and Ahlatcioglu, 2009):

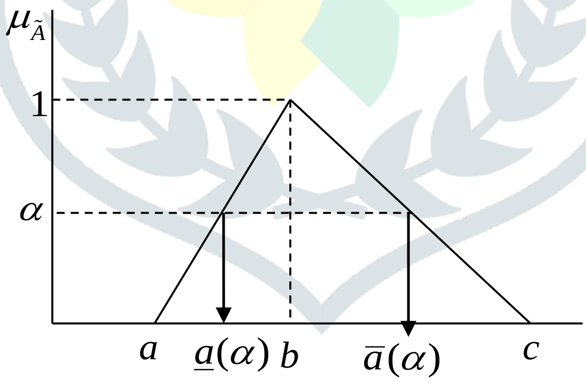
$$\mu_{\tilde{M}}(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & b < x \le c \\ \dfrac{d-x}{d-c} & c < x \le d \\ 0 & x > d \end{cases}$$

Its α-level set is: $\tilde{A}_{\alpha} = [\underline{a}(\alpha), \overline{a}(\alpha)] = [a + (b-a)\alpha, d - (d-c)\alpha].$

- **Triangular Fuzzy Number (TFN)**

When $b = c$, the TrFN transform to triangular fuzzy number (TFN), shown in **Error! Reference source not found.** denoted by $\tilde{M} = (a, b, d)$ where $a \le b \le d$ has triangular-type membership function (Huang et al. 2008):

$$\mu_{\tilde{M}}(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ \dfrac{b-x}{b-c} & b < x \le c \\ 0 & x > c \end{cases}$$



By defining the interval of confidence level α, the triangular fuzzy number can be described as: $\tilde{M}_{\alpha} = [\ \underline{a}(\alpha),\ \overline{a}(\alpha)] = [a + (b-a)\alpha, c - (c-b)\alpha].$

Definition 2.1 Let $\mathbb{R}$ be the set of real numbers and $a: \mathbb{R} \to [0, 1]$ be a membership function of a fuzzy set. Then a is said to be a fuzzy number if there is some $r_0 \in \mathbb{R}$ such that $a(r_0) = 1$; function a is quasi-concave and upper semi-continuous on $\mathbb{R}$ and closure of a set $\{r \in \mathbb{R} / a(r) > 0\}$ forms a compact set. The set of fuzzy numbers on $\mathbb{R}$ is denoted by $F(\mathbb{R})$.

To study the properties of fuzzy numbers, we define corresponding crisp (ordinary) sets.

Definition 2.2 The α −level set a α of any $a \in F(\mathbb{R})$ is a crisp set $a_{\alpha} = \{r \in \mathbb{R} / a(r) \ge \alpha\}$, for $\alpha \in (0,1]$ and the 0 −level set a_0 is defined as the closure of the set $\{r \in \mathbb{R} / a(r) > 0\}$.

Using the theorem of Goetschel and Voxman4 , we have following the characterization of a fuzzy number.

Theorem 2.1 A fuzzy number a is defined by any pair $a = (a_1(\alpha), a_2(\alpha))$ of functions $a_i : [0,1] \to \mathbb{R}$, defining the end-points of the α −level sets, satisfying the following conditions:

(i) a_1 is a bounded nondecreasing left-continuous function for $\alpha \in (0,1]$ and right-continuous at $\alpha = 0$;

(ii) a_2 is a bounded nonincreasing left-continuous function for $\alpha \in (0,1]$ and right continuous at $\alpha = 0$;

(iii) $a_1(\alpha) \leq a_2(\alpha)$ for all α.

Addition, subtraction, multiplication of two fuzzy numbers are not the same as defined for real numbers. As fuzzy numbers are defined by membership functions, their arithmetic operations can be defined using continuous operation.

Definition 2.3 Addition and scalar multiplication using their α-level sets are given as follows:

$(a + b)_\alpha = [(a+b)_1(\alpha), (a+b)_2(\alpha)]$, where $(a+b)_1(\alpha) = a_1(\alpha) + b_1(\alpha)$ and $(a+b)_2(\alpha) = a_2(\alpha) + b_2(\alpha)$

$(\lambda\, a)_\alpha = [(\lambda a)_1(\alpha), (\lambda a)_2(\alpha)]$, if λ ≥ 0 $= [(\lambda a)_2(\alpha), (\lambda a_1)(\alpha)], if\ \lambda < 0$, where $(\lambda a)_1(\alpha) = \lambda\, a_1(\alpha)$ and $(\lambda a)_2(\alpha) = \lambda\, a_2(\alpha)$.

for α-level sets $a_\alpha = [a_1(\alpha), a_2(\alpha)], b_\alpha = [b_1(\alpha), b_2(\alpha)]$, for $\alpha \in [0,1]$ and scalar $\lambda \in \mathbb{R}$.

.3 Generalized Seikkala differentiability and its properties

Rate of change of function is a derivative of that function. If we consider rate of change of imprecise or fuzzy function, we have fuzzy derivative concept. For example, some object is falling from specific height. The motion of the object after t seconds is known approximately can be expressed by fuzzy function. To find instantaneous rate of change of object with respect to time is given as fuzzy derivative at that time. Many authors have defined fuzzy derivatives mathematically in literature. In this paper, we use concept of Seikkala derivative of fuzzy function to study the fuzzy growth and decay systems.

3.1 Seikkala differentiability

Seikkala differentiability of fuzzy-valued function $y: I \to F(\mathbb{R})$ is defined as follows. The definition is adopted from Seikkala3

.**Definition 3.1** Let I be subset of $\mathbb{R}$ and y be a fuzzy-valued function defined on I. The α −level sets $y_\alpha(t) = [y_1(t,\alpha), y_2(t,\alpha)]$ for $\alpha \in [0,1]$ and $t \in I$. We assume that derivatives of $y_i(t,\alpha), i = 1,2$ exist for all $t \in I$ and for each α.

We define $(\boldsymbol{y'(t)})\boldsymbol{\alpha} = [\boldsymbol{y_1}'(\boldsymbol{t},\boldsymbol{\alpha}), \boldsymbol{y_2}'(\boldsymbol{t},\boldsymbol{\alpha})]$ for all $t \in I$, all α.

If, for each fixed $t \in I$, $\left(y'(t)\right)_\alpha$ defines the α −level set of a fuzzy number, then we say that Seikkala derivative of $y(t)$ exists at t and it is denoted by fuzzy-valued function $y'(t)$.

The Seikkala derivative involves two steps:

(1) Check both level functions are differentiable or not

(2) Check level sets of derivatives define fuzzy numbers or not.

Exponential functions are used to represent real-world applications, such as bacterial growth/decay, population growth/decline. If the initial population is imprecise or inexact then to represent the system following fuzzy functions are used. And to find rate of change of such function, we use fuzzy derivative. In particular, Seikkala derivative. So we consider some illustrations of how to find Seikkala derivatives of functions those are useful in real life situations.

The following fuzzy function is exists in fuzzy decay problem. We study the rate of change of the function using uncertain (fuzzy) derivative.

Example 3.1 Consider a fuzzy-valued function $g(t) = a\, exp(-t), t \in \mathbb{R}$ and a is a fuzzy number with α −level sets $g_\alpha(t) = [g_1(t,\alpha), g_2(t,\alpha)] = [a_1(\alpha)\, exp(-t), a_2(\alpha)\, exp(-t)]$. To check Seikkala differentiability of given fuzzy-valued function, first we check both its level functions are differentiable or not.

We see that $g_1(t,\alpha) = a_1(\alpha)\, exp(-x)\ and\ g_2(x,\alpha) = a_2(\alpha)\, exp(-t)$ are differentiable for $t \in \mathbb{R}$.

Next, we check that the level sets

(g'(t))α = [g1'(t, α), g2'(t, α)] = [-a1(α)exp(t), -a2(α)exp(t)]

define a fuzzy number for each t in R. By checking sufficient conditions for (g'(t))α to define α-level sets of fuzzy number,

(i) g1'(t, α) is an increasing function of α for each t\in R;

(ii) g2'(t, α) is a decreasing function of α for each t\in R; and

(iii) g1'(t, α) ≤ g2'(t, α) for all t \in R, we see that

$\partial g_1{}'(t,\alpha)/\partial\alpha = -a1'(\alpha)\, exp(t) < 0$ as $a1'(\alpha) > 0$ & $\partial g_1{}'(t,\alpha)/\partial\alpha = g_2{}'(t,\alpha)/\alpha = -a2'(\alpha)\, exp(t) < 0$ as a2'(α) < 0.

Therefore, Seikkala derivative of g does not exist.

We consider another example of derivative of fuzzy function which occur in uncertain periodic motion of an object.

Example 3.2 Consider a fuzzy-valued function h(t) = a sin(t), t in [0, □], where a is a fuzzy number. The α- level sets of h(t) are [a1(α)sin(t), a2(α)sin(t)]. The level function are differentiable but their derivatives h1'(t, α) = a1(α) cos(t) and h2'(t, α) = a2(α) cos(t) does not define fuzzy number for each t in [□/2, □] and hence h is not Seikkala differentiable for $t \in \frac{\pi}{2s}$

3.2 Generalized Seikkala differentiability

From Example 3.1 and 3.2, we see that not all fuzzy-valued functions are Seikkala differentiable. The generalized Seikkala derivative (gS-derivative) of a fuzzy-valued function is defined as follows:

Definition 3.2 Let I be a real interval. A fuzzy-valued function f: I → F(R) with α-level sets fα(t) = [f1(t, α), f2(t, α)], for t \in I and α \in [0, 1] is said to have generalized Seikkala derivative f ′(t) if f1(t, α) and f2(t, α) are differentiable for each t \in I and f ′α(t) = [min{f1′(t, α), f2′(t, α)}, max{ f1′(t, α), f2′(t, α)}], for all α defines a fuzzy number for each t in I.

We see that the uncertain functions defined in Example 3.1 and 3.2 exist in real life, are gS differentiable. Example 3.3 The fuzzy-valued function g(t) = a exp(-t), t in R, defined in Example 3.2. The derivatives of level functions of g(t) are g1'(t, α) = -a1(α) exp(-t) and g2'(t, α) = -a2(α) exp(-t).

By definition of gS-differentiability, α-level sets gα'(t) defined as

gα'(t) = [min{-a1(α) exp(-t), -a2(α) exp(-t)}, max{-a1(α) exp(-t), -a2(α) exp(-t)}]

which is equal to gα'(t) = [-a2(α) exp(-t), -a1(α) exp(-t)]
as - a2(α) ≤ - a1(α) and exp(-t) ≥ 0 for all t.
Therefore, g is gS-differentiable with derivative g′(t) = -a exp(-x).
Example 3.4 The fuzzy-valued function h(t) defined in Example 3.3 is gS-differentiable with derivative h′(t) = a cos(t). The α-level sets of h′(t) are
[a1(α) cos(t), a2(α) cos(t)] for t \in [0, □/2] and [a2(α) cos(t), a1(α) cos(t)] for x \in (□/2, □].

4. Runge – Kutta 5th order method

Let us consider the first order simultaneous differential equation

$$\frac{dx}{dt} = f(t,x,y)\ \&\ \frac{dy}{dt} = g(t,x,y)$$

with initial conditions

$$x(t_0) = x_0, y(t_0) = y_0$$

General form for numerical solution of ordinary differential equation

$$x(t_{n+1}) = x(t_n) + w_1k_1 + w_2k_2 + w_3k_3 + w_4k_4 + w_5k_5$$
$$k_1 = hf(t_n, x_n)$$
$$k_2 = hf(t_n + c_2h, x_n + a_{21}k_1)$$
$$k_3 = hf(t_n + c_3h, x_n + a_{31}ka_{32}k_2)$$
$$k_4 = hf(t_n + c_4h, x_n + a_{41}k + a_{42}k_2 + a_{43}k_3,)$$
$$k_5 = hf(t_n + c_5h, x_n + a_{51}k + a_{52}k_2 + a_{53}k_3 + a_{54}k_4)$$

Utilising Taylor's series expansion techniques, Runge Kutta 5th order is given by

$$x_{n+1} = x_n + \frac{k_1 + 4k_4 + k_5}{6}$$
$$y_{n+1} = x_n + \frac{l_1 + 4l_4 + l_5}{6}$$

$$k_1 = hf(t_n, x_n, y_n)$$
$$k_2 = hf(t_n + \frac{1}{3}h, x_n + \frac{1}{3}k_1, y_n + \frac{l_1}{3})$$
$$k_3 = hf(t_n + \frac{1}{3}h, x_n + \frac{1}{6}k_1 + \frac{1}{6}k_2, y_n + \frac{1}{6}l_1 + \frac{1}{6}l_2)$$
$$k_4 = hf(t_n + \frac{1}{3}h, x_n + \frac{1}{8}k_1 + \frac{3}{8}k_3, x_n + \frac{1}{8}l_1 + \frac{3}{8}l_3)$$
$$k_5 = hf(t_n + h, x_n + \frac{1}{2}k_1 - \frac{3}{8}k_2 + 2k_4, x_n + \frac{1}{2}l_1 - \frac{3}{8}l_2 + 2l_4)$$

$$l_1 = hg(t_n, x_n, y_n)$$
$$l_2 = hg(t_n + \frac{1}{3}h, x_n + \frac{1}{3}k_1, y_n + \frac{l_1}{3})$$
$$l_3 = hg(t_n + \frac{1}{3}h, x_n + \frac{1}{6}k_1 + \frac{1}{6}k_2, y_n + \frac{1}{6}l_1 + \frac{1}{6}l_2)$$
$$l_4 = hg(t_n + \frac{1}{3}h, x_n + \frac{1}{8}k_1 + \frac{3}{8}k_3, x_n + \frac{1}{8}l_1 + \frac{3}{8}l_3)$$
$$l_5 = hg(t_n + h, x_n + \frac{1}{2}k_1 - \frac{3}{8}k_2 + 2k_4, x_n + \frac{1}{2}l_1 - \frac{3}{8}l_2 + 2l_4)$$

Example 4.1

$$\frac{dx}{dt} = f(t,x,y) = -4x + 5y$$
$$\frac{dy}{dt} = g(t,x,y) = 8y - 6z$$

Fuzzy initial conditions $x(0) = (7.6 + 0.4r, 8.3 - 0.2r)$ and $y(0) = (3.8 + 0.2r, 4.3 - 0.3r), 0 \le r \le 1$

x and y are fuzzy numbers, $\tilde{x} = (x_1\ , x_2\ ), \tilde{y} = (y_1 y_2)$

$$\frac{d\tilde{x}}{dt} = f(t, \tilde{x}, y) = -4\tilde{x} + 5y$$
$$\frac{d\tilde{y}}{dt} = g(t, x, \tilde{y}) = 8\tilde{x} - 6\tilde{y}$$

Exact solution is $\underline{X}(t,r) = 3\underline{x}_1e^t + 3\underline{x}_2e^{-3t}, \underline{Y}(t,r) = 3\underline{y}_1e^t + 3\underline{y}_2e^{-3t}$

4.1 Comparison of Results

			Numerical Solution				Exact Solution			
n	r	h	Lyn+1	Uyn+1	Lzn+1	Uzn+1	LYn+1	UYn+1	LZn+1	UZn+1
0	0	0.1	7.3133407	7.9555886	6.4918778	7.1384598	7.6744233	9.1331384	7.5151047	8.8941604
0	0.1	0.1	7.3518319	7.929855	6.5260456	7.1079694	7.7473591	9.0602026	7.5840575	8.8252076
0	0.2	0.1	7.3903232	7.9041215	6.5602133	7.0774789	7.8202948	8.9872668	7.6530103	8.7562549
0	0.3	0.1	7.4288145	7.878388	6.5943811	7.0469885	7.8932306	8.9143311	7.721963	8.6873021
0	0.4	0.1	7.4673057	7.8526545	6.6285489	7.0164981	7.9661663	8.8413953	7.7909158	8.6183493
0	0.5	0.1	7.505797	7.8269209	6.6627167	6.9860077	8.0391021	8.7684596	7.8598686	8.5493965
0	0.6	0.1	7.5442883	7.8011874	6.6968844	6.9555172	8.1120378	8.6955238	7.9288214	8.4804437
0	0.7	0.1	7.5827795	7.7754539	6.7310522	6.9250268	8.1849736	8.6225881	7.9977742	8.4114909
0	0.8	0.1	7.6212708	7.7497204	6.76522	6.8945364	8.2579093	8.5496523	8.066727	8.3425381
0	0.9	0.1	7.6597621	7.7239869	6.7993878	6.864046	8.3308451	8.4767166	8.1356798	8.2735853
0	1	0.1	7.6982533	7.6982533	6.8335556	6.8335556	8.4037808	8.4037808	8.2046326	8.2046326

5. Conclusion

In this paper we have found the iterative solution of first order simultaneous fuzzy differential equation using fifth order Runge-Kutta Merson method.

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