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A New Axioms of Abelian Groups

Without Associativity by Using Symmetric Groupoid

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Abstract **:** In this work, the concept of the symmetric groupoid is introduced and some its properties are investigated. We use symmetric groupoid to give the new axioms of commutative group which does not necessarily has associative law. Futhermore, we connect of this non-associative structure with other algebraic structures to make it interesting and useful.

Keywords**:** commutative group, symmetric groupoid, LA-semigroup.
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I. Introduction

A groupoid is a basic algebraic structure of group which have been widely studied and used in many fields, such as physics, chemistry, engineering, and so on. Group is a very good mathematical tool to describe the symmety of the nature [4, 11]. It is well-known that a groupoid G is an abelian group or commutative group if

(i) $a(bc) = (ab)c$ for all $a, b, c \in G$,

(ii) for each $a, b \in G$, there exist $x, y \in G$ such that $ax = b$ and $ya = b$ or the equation $ax = b$ and $ya = b$ are solvable for all $a, b \in G$,

(iii) $ab = ba$ for all $a, b \in G$.

As we all know, the concept of group is a useful tool to characterize symmetry. In 1971 [13], R. L. Duncan and Hilda F. Duncan introduced the notion of symmetric groupoids. A groupoid is called left [right] symmetric if all its elements satisfy the left [right] symmetric law : $a(bc) = (ba)c$ [$a(bc) = (ac)b$] and is called symmetric if it is both left and right symmetric. This algebraic structure is midway between a groupoid and a commutative semigroup. In 1978, W. Yongpitayapapong studied of symmetric groupoids and rings and also obtained several characterizations and properties [14]. In this paper, we study some algebraic structures of symmetric groupoids to define a new axioms of commutative group on this algebraic structure. Futhermore, we connect of this non-associative structure with other algebraic structures to make it interesting and useful.

II. Preliminaries

We first present propositions on symmetric groupoids which most will be used in the next.

Definition 2.1. *A groupoid S is a medial if $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in S$ and a paramedial if $(ab)(cd) = (dc)(ba)$ for all $a, b, c, d \in S$.*

As a direct consequence of the above definitions, we have the following propositions.

Proposition 2.2. *Every symmetric groupoid is medial.*

Proposition 2.3. *Every symmetric groupoid is paramedial.*

Theorem 2.4. *[14] Every commutative semigroup is symmetric groupoid.*

Example 2.5. The left symmetric is not neccessary to be right symmetric groupoid. Let $S = \{a, b, c\}$ and $*$ be a binary operation on S defined by the following table,

$*$	a	b	c
a	a	b	c
b	a	b	c
c	a	b	c

It is clear that S is left symmetric groupoid, but S is not right symetric groupoid. Because $a(bc) = c \neq b = (ac)b$.

Example 2.6. The right symmetric is not neccessary to be left symmetric groupoid. Let $S = \{a, b, c\}$ and $*$ be a binary operation on S defined by the following table,

$*$	a	b	c
a	a	a	a
b	b	b	b
c	c	c	c

It is clear that *S* is right symmetric groupoid, but *S* is not left symetric groupoid. Because $a(bb) = a \neq b = (ba)b$.

Theorem 2.7. *[14] Every left (right) symmetric commutative groupoid is commutative semigroup.*
Theorem 2.8. *[14] Every left (right) symmetric groupoid with right (left) identity is commutative semigroup.*
Theorem 2.9. *[14] Every right (left) cancellative symmetric groupoid is commutative semigroup.*

III. Main Results

The next results describe the great important relationship between symmetric groupoid and group.

Theorem 3.1. *Every left symmetric groupoid S satisfying for each $a, b \in S$, there exist $x \in S$ such that $xa = b$ is a commutative group.*

Proof. Let *S* be a left symmetric groupoid satisfying for $a, b \in S$, there exist $x \in S$ such that $xa = b$. Consider

$ab = a(xa)$ $(b = xa)$
$= (xa)a$ (left symmetric)
$= ba$

Hence *S* is commutative groupoid. By theorem 2.7, we have *S* is commutative semigroup. From the assumtion and commutativity, for all $a, b \in S$, there exist $x \in S$ such that $ax = xa = b$. We conclude that *S* is a commutative group. □

Example 3.2. Let $S = \{1, 2, 3\}$ and $*$ be a binary operation on *S* defined by $a * b = a$ for all $a, b \in S$. We have the following table,

$*$	1	2	3
1	1	1	1
2	2	2	2
3	3	3	3

We see that *S* is a left zero semigroup satisfying for $a, b \in S$, there exist $x \in S$ such that $xa = b$. But there is no $x \in S$ such that the equation $3x = 2$ hold. So *S* is not group.

Theorem 3.3. *Every right symmetric groupoid S satisfying for each $a, b \in S$, there exist $x \in S$ such that $ax = b$ is a commutative group.*

Proof. Let *S* be a right symmetric groupoid satisfying for $a, b \in S$, there exist $x \in S$ such that $ax = b$. Consider

$ab = a(ax)$ $(b = ax)$
$= (ax)a$ (right symmetric)
$= ba$

Hence *S* is commutative groupoid. By theorem 2.7, we have *S* is commutative semigroup. From the assumtion and commutativity, for all $a, b \in S$, there exist $x \in S$ such that $xa = ax = b$. We conclude that *S* is a commutative group. □

Example 3.4. Let $S = \{1, 2, 3\}$ and $*$ be a binary operation on *S* defined by $a * b = b$ for all $a, b \in S$. We have the following table,

$*$	1	2	3
1	1	2	3
2	1	2	3
3	1	2	3

We see that *S* is a right zero semigroup satisfying for $a, b \in S$, there exist $x \in S$ such that $ax = b$. But there is no $x \in S$ such that the equation x3 = 2 hold. So *S* is not group.

As a direct consequence of Theorem 2.8 and commutativity, it is easy to prove the following corollary and the next theorem is equivalent to the usual group.

Corollary 3.5. *Every left (right) symmetric groupoid with right (left) identity is commutative monoid*.
Theorem 3.6. *[9] Let S be a semigroup, then the following statements are equivalent.*
1. S is group.
2. S has a left identity e and For all $a \in S$, there is $b \in S$ such that $ba = e$.

3. S has a right identity e and For all a ∈ S, there is b ∈ S such that ab = e.

In Example 3.2, *S* is a semigroup with right identity *e* and For all *a* ∈ *S*, there is *b* ∈ *S* such that *ba* = *e*. But *S* is not group.
In Example 3.4, *S* is a semigroup with left identity *e* and For all *a* ∈ *S*, there is *b* ∈ *S* such that *ab* = *e*. But *S* is not group.
If we replace semigroup in theorem 3.6 by left(right) symmetric groupoid, we have the following importance results.

Theorem 3.7. *Let S be a left symmetric groupoid, then the following statements are equivalent.*
1. S is commutative group.
2. S has a right identity e and For all a ∈ S, there is b ∈ S such that ab = e.
3. S has a right identity e and For all a ∈ S, there is b ∈ S such that ba = e.

Proof. Let *S* be a left symmetric groupoid.
(1) ⇒ (2) Obviousely by theorem 3.6.
(2) ⇒ (3) Suppose *S* has a right identity *e* and For all *a* ∈ *S*, there is *b* ∈ *S* such that *ab* = *e*. By Corollary 3.5, *S* is commutative monoid. So for all *a* ∈ *S*, there is *b* ∈ *S* such that *ba* = *ab* = *e*.
(3) ⇒ (1) Suppose *S* has a right identity *e* and For all *a* ∈ *S*, there is *b* ∈ *S* such that *ba* = *e*. By Corollary 3.5, *S* is commutative monoid. And for all *a* ∈ *S*, there is *b* ∈ *S* such that *ab* = *ba* = *e*. Therefore *S* is commutative group. □

Theorem 3.8. *Let S be a right symmetric groupoid, then the following statements are equivalent.*
1. S is commutative group
2. S has a left identity e and For all a ∈ S, there is b ∈ S such that ba = e.
3. S has a left identity e and For all a ∈ S, there is b ∈ S such that ab = e.

Proof. Let *S* be a right symmetric groupoid.
(1) ⇒ (2) Obviousely by theorem 3.6.
(2) ⇒ (3) Suppose *S* has a left identity *e* and For all *a* ∈ *S*, there is *b* ∈ *S* such that *ab* = *e*. By Corollary 3.5, *S* is commutative monoid. So for all *a* ∈ *S*, there is *b* ∈ *S* such that *ab* = *ba* = *e*.
(3) ⇒ (1) Suppose *S* has a left identity *e* and For all *a* ∈ *S*, there is *b* ∈ *S* such that *ba* = *e*. By Corollary 3.5, *S* is commutative monoid. And for all *a* ∈ *S*, there is *b* ∈ *S* such that *ba* = *ab* = *e*. Therefore *S* is commutative group. □

Definition 3.9. *A semigroup S is said to be an regular semigroup if every element a ∈ S, there is b ∈ S such that a = aba. It is easy to prove that every group is regular semigroup.*

Lemma 3.10. *Let S be a semigroup and for all a ∈ S, there is a unique b ∈ S such that a = aba, then S has exactly one idempotent element.*

Proof. Let *S* be a semigroup and for all *a* ∈ *S*, there is a unique *b* ∈ *S* such that *a* = *aba*. Since *ab* = *abab* = $(ab)^2$, we obtain *x* = *ab* is an idempotent element in S. Suppose that $y^2 = y$. To show that *x* = *y*. Consider *xy* ∈ *S*. There exists unique *z* ∈ *S* such that *xy* = *xyzxy*. Consider

$$\begin{aligned} xy &= xyzxy \\ &= xy^2zx^2y \\ &= xy(yzx)xy \end{aligned}$$

By assumtion, we have *z* = *yzx*. Since *yz* = *y*(*yzx*) = *yzx* = *z and zx* = (*yzx*)*x* = *yzx* = *z*, we have

$$\begin{aligned} xy &= x(yzx)y \\ &= xzy \\ &= x(yz)y \end{aligned}$$

So *xy* = *xyzxy* = (*xy*)*zxy* = (*xyzy*)*zxy* = *xy*(*zyz*)*xy*. By assumtion, we have *z* = *zyz*. From $z^2 = zz = z(yz) = z$ and $z = z^2 = zz = (zx)z$, we obtain *x* = *y*. We conclude that *S* has exactly one idempotent element. □

Theorem 3.11. *[9] Let S be a regular semigroup, then S has exactly one idempotent element if and only if S is a group.*
Theorem 3.12. *Let S be a semigroup, then the following statements are equivalent.*
1. For all a ∈ S, there is a unique b ∈ S such that a = aba.
2. S is a regular semigroup and has exactly one idempotent element.
3. S is a group.

Example 3.13. Let *S* be a non-empty set and *P(S)* the set of all subset of *S*. It is well-khown that *(P(S), ∪)* is a commutative monoid with identity ∅ and also have that *(P(S), ∪)* is regular semigroup because for each *A* ⊆ *S* is an idempotent of *P(S)*. Moreover, for each ∅ ≠ *A* ⊆ *S*, there are ∅ and *A* such that *A* = *A* ∪ *A* ∪ *A* and *A* = *A* ∪ ∅ ∪ *A*. So *(P(S), ∪)* is not group.

As a direct consequence of Theorem 3.1, 3.3 and Theorem 3.9, we get the following corollary.

Corollary 3.14. *Let S be a groupoid, then the following statements are equivalent.*
1. S is a left symmetric and for each a, b ∈ S, there exist x ∈ S such that xa = b.
2. S is a right symmetric and for each a, b ∈ S, there exist x ∈ S such that ax = b.
3. S is a commutative group.
4. S is a commutative semigroup and for all a ∈ S, there is a unique b ∈ S such that a = aba .

5. S is a commutative semigroup and for each a, b ∈ S, there exist x ∈ S such that xa = b.
6. S is a commutative semigroup and for each a, b ∈ S, there exist x ∈ S such that ax = b.

Definition 3.15. *[10] A groupoid S is said to be an AG-groupoid or LA-semigroup if (ab)c = (cb)a for all a, b, c ∈ S and a RA-semigroup if a(bc) = c(ba) for all a, b, c ∈ S. An AG-groupoid S satisfying the identity (ab)c = b(ac) is called AG*-groupoid.*

Theorem 3.16. *Let S be a groupoid with right identity, then the following statements are equivalent.*
1. S is left symmetric groupoid.
2. S is commutative semigroup.
3. S is AG-groupoid.

Proof. Let *S* be a groupoid with right identity *e* and *a, b, c ∈ S.*
(1) ⇒ (2) Obviousely by theorem 2.8.
(2) ⇒ (3) Let *S* be commutative semigroup. We have *(ab)c = a(bc) = (bc)a = (cb)a*. So *S* is AG-groupoid.
(3) ⇒ (1) Let *S* be AG-groupoid. Since *e* is right identity, we obtain *ab = (ae)b = (be)a = ba*. We have *(ab)c = (ba)c = (ca)b = b(ca) = b(ac)*. Therefore *S* is left symmetric groupoid.
In Example 3.2, *S* is a semigroup with right identity. But S is not both AG-groupoid and left symmetric groupoid. Because (12)3 = 1 ≠ 3 = (32)1 and 1(21) = 1 ≠ 2 = (21)1. □

In Example 3.4, S is a left symmetric groupoid with left identity. But *S* is not both AG-groupoid and commutative semigroup. Because (12)3 = 3 ≠ 1 = (32)1 and 21 = 1 ≠ 2 = 12.

Example 3.17. Let *S = {a, b, c}* and ∗ be a binary operation on *S* defined by the following table,

∗	a	b	c
a	a	a	a
b	a	a	a
c	a	b	a

It is clear that *S* is AG-groupoid. But *S* is not commutative semigroup. Because *c(cb) = b ≠ a = (cc)b* and *cb = b ≠ a = bc*. Moreover *S* is not RA-semigroup because *c(cb) = b ≠ a = b(cc)*.

Theorem 3.18. *Let S be a groupoid with left identity, then the following statements are equivalent.*
1. S is right symmetric groupoid.
2. S is commutative semigroup.
3. S is RA-semigroup.

Proof. Let *S* be a groupoid with left identity *e* and *a, b, c ∈ S.*
(1) ⇒ (2) Obviousely by theorem 2.8.
(2) ⇒ (3) Let *S* be commutative semigroup. We have *a(bc) = (ab)c = c(ab) = c(ba).* So *S* is RA-semigroup.
(3) ⇒ (1) Let *S* be RA-semigroup. Since *e* is left identity, we obtain *ab = a(eb) = b(ea) = ba*. We have *a(bc) = a(cb) = b(ca) = (ca)b = (ac)b*. Therefore *S* is right symmetric groupoid. □

In Example 3.4, *S* is a semigroup with left identity. But *S* is not both RA-semigroup and right symmetric groupoid. Because (12)3 = 1 ≠ 3 = (32)1 and 1(21) = 1 ≠ 2 = (21)1.
In Example 3.2, *S* is a right symmetric with right identity. But *S* is not both RA-semigroup and commutative semigroup. Because 1(22) = 1 ≠ 2 = 2(21) and 12 = 1 ≠ 2 = 21.

Example 3.19. Let *S = {a, b, c}* and ∗ be a binary operation on *S* defined by the following table,

∗	a	b	c
a	a	a	a
b	a	a	b
c	a	a	a

It is clear that *S* is RA-semigroup, but *S* is not commutative semigroup. Because *b(cc) = a ≠ b = (bc)c* and *cb = a ≠ b = bc*. Moreover *S* is not AG-groupoid because *(bc)c = b ≠ a = (cc)b*.

Theorem 3.20. *Let S be a commutative groupoid, then the following statements are equivalent.*
1. S is semigroup.
2. S is left symmetric groupoid.
3. S is AG-groupoid.
4. S is right symmetric groupoid.

5. S is RA-semigroup.

Proof. Let *S* be a commutative groupoid and *a, b, c* $\in S$.

(1) ⇒ (2) Obvious.

(2) ⇒ (3) Let *S* be left symmetric. We have *(ab)c = b(ac) = b(ca) = (cb)a*. So *S* is AG-groupoid.

(3) ⇒ (4) Let *S* be AG-groupoid. We have *a(bc) = (bc)a = (ac)b*. Thus *S* is right symmetric.

(4) ⇒ (5) Let *S* be right symmetric. We have *a(bc) = (ac)b = (ca)b = c(ba)*. Thus *S* is RA-semigroup.

(5) ⇒ (1) Let *S* be RA-semigroup. We have *a(bc) = c(ba) = (ba)c = (ab)c*. Thus *S* is semigroup. □

As a direct consequence of Theorem 2.8 and above theorem, it is easy to prove the following corollary.

Corollary 3.21. *Every left (right) symmetric groupoid with right (left) identity is RA(LA)-semigroup.*

Corollary 3.22. *Let S be an AG-groupoid, then S is left symmetric groupoid if and only if S is AG*-groupoid.*

Proof. Let *S* be an AG-groupoid and *a, b, c* $\in S$.

(⇒) Suppose that *S* is a left symmetric groupoid. Then *(ab)c = b(ac).* So *S* is AG*-groupoid.

(⇐) Suppose *S* is AG*-groupoid. Then *a(bc) = (ba)c*. Therefore *S* is left symmetric groupoid. □

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