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NONSTANDARD TECHNIQUE TO EXTEND A LINEAR FUNCTIONAL FROM A SUBSPACE TO THE WHOLE SPACE

Pawan Kumar Karn
Department of Mathematics, Mahendra Morang Adarsha, Multiple Campus, Biratnagar, Tribhuvan University, Nepal.
Email: infopawankarn@gmail.com

Abstract: This paper discusses the nonstandard technique to extend a linear functional from a subspace to the whole space in such a manner that various properties of the functional are preserved and discusses a theorem which appears as problem in Robinson's model theory. It also associates the concept of Banach limit with the functional standard (i.e., st).

Key words:- Linear space, linear functional, standard sequence, Banach limit.

INTRODUCTION

This paper is going to discuss a class of spaces that are endowed with both a topological and an algebraic structure. First, it gives some useful definitions which are as follows:

Definitions (1)(6):

(a)Linear space of standard sequences:- A set X of standard sequences $\{x_n\}$ is called a linear space over reals if there exists a function + on $X \times X$ to X and a function . on $R \times X$ to X that satisfy the following conditions:

i) $\{x_n\}+\{y_n\}=\{y_n\}+\{x_n\}$

ii) $(\{x_n\}+\{y_n\})+\{z_n\}=\{x_n\}+(\{y_n\}+\{z_n\})$

iii) There is a unique standard sequence $\{\theta\}$ in X such that $\{x_n\}+\{\theta\}=\{x_n\}$ for all $\{x_n\}\in X$.

iv) $\lambda(\{x_n\}+\{y_n\})=\lambda\{x_n\}+\lambda\{y_n\};\lambda\in R;\{x_n\},\{y_n\}\in X$

v) $(\lambda+\mu)\{x_n\}=\lambda\{x_n\}+\mu\{x_n\};\lambda,\mu\in R;\{x_n\}\in X$

vi) $\lambda(\mu\{x_n\})=(\lambda\mu)\{x_n\};\lambda,\mu\in R;\{x_n\}\in X$

vii) $0.\{x_n\}=\{\theta\};\ 1.\{x_n\}=\{x_n\}$

Here + is called addition and . is called multiplication by scalars.

(b)Linear operator:- A mapping A of linear space X into a linear space Y is called linear operator if $A(\alpha_n\{x_n\}+\beta_n\{y_n\})=\alpha_n A\{x_n\}+\beta_n A\{y_n\}$ for all $\{x_n\},\{y_n\}\in X$ for all reals α_n,β_n.

(c)Linear Functional:- A linear functional f^* on a linear space X is a linear operator from X to the space R^* (i.e., ultra real number system). Thus a linear functional is a ultra real valued function f^* on X such that

$f^*(\alpha\{x_n\}+\beta\{y_n\})=\alpha f^*(\{x_n\})+\beta f^*(\{y_n\})$.

Now we proceed to apply the nonstandard technique in proving a few theorems of Functional Analysis.

Theorem (1):-(Hahn-Banach)(2)If p be a semi linear functional defined on a real linear space X and f be a real linear functional defined on a subspace S of X dominated by p (i.e., $f(x)\le p(x)$ for all $x\in S$).Then there exists a real linear functional F on X such that (i) $F(x)=f(x)$ for all $x\in S$, and (ii) $F(x)\le p(x)$ for all $x\in X$.

Proof:-Given that S be a subspace of linear space X .Let $I=\{v\}$ be the index set of the family of all linear functional $\{f_v\}$ which are defined on linear subspaces of X , each of which contains S and is an extension of f so that

(i) $f_v(x)=f(x)$ for all $x\in S, v\in I$,

(ii) $Df_v \supseteq Df$ for all $v\in I$, where $D\phi$ means 'domain of ϕ', and let

(iii) $f_v(x)\le p(x)$ for all $x\in X\bigcap Df_{v.}$

For any $x\in X$, let I_x denote the subset of I for which $x\in Df_v$,

i.e., let $I_x=\{v: v\in I. x\in Df_v\}$

Let $\alpha_1,\alpha_{2,}...,\alpha_n\in I$ and let $x_1x_2,...,x_n$ be distinct elements of $X-S$.

Let f_{α_1} be an extension of f such that $x_1\in Df_{\alpha_1}$ and $f_{\alpha_1}(x)=f(x)$ for all $x\in S$,

then $\alpha_1\in I_{x_1}$ .Similarly, let f_{α_2} be an extension of f_{α_1} so that $x_1,x_2\in Df_{\alpha_2}$.

Then $\alpha_2\in I_{x_2}$ and also $\alpha_2\in I_{x_1}$ .Continuing this process of extension of f up to f_{α_n} such that $x_n\in Df_{\alpha_n}$,we get that $\alpha_1,\alpha_2,...,\alpha_{n-1},\alpha_n\in I_{x_1}$,

$$\alpha_2,...,\alpha_{n-1},\alpha_n\in I_{x_2},$$
$$\alpha_{n-1,}\alpha_n\in I_{n-1},$$
$$\alpha_n\in I_{x_n},$$

and $f_{\alpha_r}(x)=f(x)$ for all $x\in S$ and $r=1,2,...,n.$

Thus we see that the family $\{I_x: x\in X\}$ of subset of I has finite intersection property viz. that if $x_i(i=1,2,...,n)\in X-S$ then $I_{x_1}\bigcap I_{x_2}\bigcap...\bigcap I_{x_n}$ is a non-empty subset of I and hence these subsets form a filter $\Im$ on I . But for every filter there exists an ultrafilter. Let U be the ultrafilter on I which contains the family of subsets $\{I_x: x\in X-S\}$ of I and $U\supset\Im$. Then the ultrapower of R with respect to I and U is given by $R_U^I=\{\phi:\phi_v\in R \ \ for\, all\ \ v\in I\}$.

Equality, sum and product of elements of R_U^I may be defined in the usual way as in case of R*. Let us now define a mapping $f^*: X\to R_U^I$ so that $f^*(x)\in R_U^I$ and is determined by $(f^*(x))_v=f_v(x)$ for all $v\in I_x$.

Then we have (i) $f^*(x)=f(x)$ for all $x\in S$, since $f_v(x)=f(x)$ for all $x\in S$ and for all $v\in I$; and (ii) $f^*(x)\le p(x)$ for all $x\in S$, since $f_v(x)\le p(x)$ for all $x\in S$ and for all $v\in I$.

Now $(f^*(-x))_v=f_v(-x)=-f_v(x)$, since f_v is a linear functional.

$$=-(f^*(x))_v \text{ for all } v\in I_x\in U$$

$\therefore f^*(-x)=-f^*(x)$. Also by pitting –$x$ for x in (ii), we have $f^*(-x)\le p(-x)$

$\therefore -f^*(x)\le p(-x)$

i.e. $f^*(x)\ge -p(-x)$ which combined with (ii) gives (iii) $--p(-x)\le f^*(x)\le p(x)$.

From (iii) it is clear that $f^*(x)$ is bounded and so finite in R_U^I .

Let $F(x)=st(f^*(x))$,

Then $F(x)$ is a real linear functional and from (i) and (ii), we have, $F(x)=f(x)$ for all $x\in S$, and $F(x)\le p(x)$ for all $x\in X$.

Now it proceeds to prove the following theorem which appears as a problem in Robinson's model theory **(5)**

Theorem(2):-If X be a linear space of bounded standard sequence $\{x_n\}$ and Y be the subspace of X consisting of all the convergent sequences, then the ordinary linear functional 'lim' on Y may be extended to a linear functional 'Lim' on X so that

Lim on X =lim on Y

Proof:- Since $\{x_n\}\in X$ is a bounded sequence , so $\{x_n\}$ is finite . Hence there exists a unique standard element viz. $st(\{x_n\})$.

We define $\text{Lim}\{x_n\} = st(\{x_n\})$ for all $\{x_n\} \in X$.

Since st is a linear functional, it follows that Lim is a linear functional on X. When $\{x_n\} \in Y$, $\{x_n\}$ is convergent and has a single limit so that, by definition of st , we have $\lim\{x_n\} = st(\{x_n\})$.

Also $\text{Lim}\{x_n\} = st(\{x_n\})$

$\therefore$ $\text{Lim}\{x_n\} = \lim\{x_n\}$ for all $\{x_n\} \in Y$

i.e., Lim on X =lim on Y

This theorem is valid for real as well as complex spaces X, Y. Lim is real or complex linear functional according as X, Y are real or complex.

Now it associates the concept of Banach limit with the functional st and for this it first define Banach limit.

Definition (2)

Banach limit(3) Banach limit is a linear functional L defined on a linear space X of all bounded sequences $x = \{x_n\}$ of real numbers such that

(a) $L(x) \geq 0$ if $x_n \geq 0$ for all x,

(b) $L(x) = L(x_2, x_3, ..., x_n, ...)$; and

(c) $L(x) = k$ if $x = (k, k, k, ...)$.

Now it proceeds to prove the following:

Theorem (3)(4) st a Banach limit on P^* (class of finite elements of R^*) .

Proof:- Let X be the linear space of all bounded sequences $\{x_n\}$ of real numbers. Then $\{x_n\}$ is finite and has a real standard part $st(x)$, where $x = (x_1, x_2, ..., x_n, ...)$. It has already been proved that st is a linear functional. In order to establish that st is a Banach limit, we have to show that

(a) $st(x) \geq 0$ if $x_n \geq 0$ for all x,

(b) $st(x) = st(x_2, x_3, ..., x_n, ...)$; and

(c) $st(x) = k$ if $x = (k, k, k, ...)$

We see that $x \in P^*$ and $st(x) = $ some $x_n \in X$. If $x_n \geq 0$ for all x, then it follows that $st(x) \geq 0$. Thus (a) is established.

Let w be an infinite natural number such that $w(n) = n + 1$.

Then $x_w = (x_2, x_3, ..., x_n, ...) \subseteq x$. Let $st(x_w) = s$.

Then $\{n : x_{w(n)} \in V_s\} \in U$, the ultrafilter on N.

$\therefore \{n : x_n \in V_s\} \in U$, and hence $st(x) = s = st(x)$ which proves (b).

If $x = (k, k, k, ...)$, x is a convergent sequence and so $st(x) = \lim(k, k, k, ...) = k$ which proves the theorem.

The theorem which we shall prove now concerns application of nonstandard technique to Measure Theory and Boolean Algebra and the same has already been proved by Luxemberg**.(1)**

Theorem(4): (Nikodym): For every Boolean Algebra X, there exists an ultrareal structure ζ^* such that X admits a strictly positive and finitely additive ζ^*-valued measure m^*.

Proof:- Let the Boolean Algebra X have two two distinct elements 0 and 1 along with other elements $x, y, ...$ and let the three operations viz. $\bigcup$ (union), $\bigcap$ (intersection) and $\wp$ (complementation) be defined on X.

Let $I = \{v\}$ be the index set of the family of real valued measures (m_v) on X such that $m_v(I) = 1$ for all $v \in I$. Since there exists a homomorphism $h : X \xrightarrow{onto} \{0,1\}$ such that $h(x) = 1$ for every non zero $x \in X$, we conclude that I is nonempty and the family of subsets $\{I_x\}$ of I where $I_x = \{v : v \in I . m_v(x) \neq 0\}$ has finite intersection property.

Associating these subsets I_x with their supersets in Z, we have a filter and ultimately an ultrafilter , Say U on I . Then we can construct an ultra real structure. $\zeta^* = R_U^I = \{f : f_v \in R$ for all $v \in I\}$ and define equality, sum and product in ζ^* as they are defined in R^* .

Let us now define a mapping **(7)** $m^*: X \to \zeta^*$ such that $m^*(x) \in \zeta^*$ for $x \in X$ and is determined by $(m^*(x))_v = m_v(x)$ for all $x \neq 0$, all $v \in I_x$ and $m^*(0) = 0$.

Since each m_v is a real valued measure function, we have for all $v \in I$ $m_v(x \bigcup y) = m_v(x) + m_v(y)$ for all $x, y \in X$ such that $x \bigcap y = 0$. Hence, we have

(i) $m^*(x \bigcup y) = m_v(x) + m_v(y)$ for all $x, y \in X$ such that $x \bigcap y = 0$. Also, we have (ii) $m^*(0) = 0$ iff $x = 0$, from definition .

The theorem is established in virtue of (i) and (ii).

Conclusion: Above discussion shows that the nonstandard technique to extend a linear functional from a subspace to the whole space preserving the properties of functional and discussed a theorem which appears as problem in Robinsion ‘s Model Theory.

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