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Pythagorean Triples

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Abstract

A Pythagorean triple is a group of three integers (*x*, *y*, *z*) such that $x^2 + y^2 = z^2$. These triples are named after the ancient Greek mathematician Pythagoras, who is credited with the Pythagorean theorem. The theorem states that in a right-angled triangle, the square of the length of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the lengths of the other two sides.

Keywords: Pythagorean Theorem, primitive Pythagorean triples.

1. Introduction

A Pythagorean triple is a set of three integers (x, y, z) (x, y, z) that satisfy the equation $z^2 = x^2 + y^2$. Notably, the knowledge of generating such triples predates Pythagoras by about 1500 years. Civilizations such as the Babylonians, who thrived between the Tigris and Euphrates rivers, as well as those along the Nile River, like the ancient Hindus and Chinese, had already discovered methods for producing these triples. An example is found on a clay tablet from the Plimpton collection at Columbia University, USA. Known as Plimpton 322, dating from 1900-1600 BCE and acquired by G. Plimpton in 1922 before being donated to Columbia University in the 1930s, this tablet was studied by researchers Neugebauer and Sachs. In 1945, they published their interpretation, revealing that the tablet lists Pythagorean triples, showing calculations of equations of the form $z^2 = x^2 + y^2$ attributed to the Babylonians.

While the Babylonians, Indians, and Chinese applied these principles long before Pythagoras, it was the Pythagoreans who formalized the study, discovering significant results such as: "any odd integer is a member of a primitive Pythagorean triple." They provided the first formal proof of this theorem and also demonstrated its converse, establishing that if the sides of a triangle satisfy the equation, then the triangle is a right triangle.

The Pythagorean triples or the Pythagorean Theorem represents one of the most important and widely recognized mathematical relationships. Its utility and popularity span almost all civilizations, attracting significant names, attention, curiosity, and evidence over the centuries. Mathematicians and non-mathematicians alike have admired this theorem, though it has also perplexed many because, unlike other theorems, there is no immediately intuitive reason why the squares on the sides of a right triangle (the hypotenuse and the legs) are so closely linked. This is why Kepler (1571-1630) regarded it as a golden jewel of mathematics. It is perhaps one of the earliest known mathematical proofs in the history of numbers.

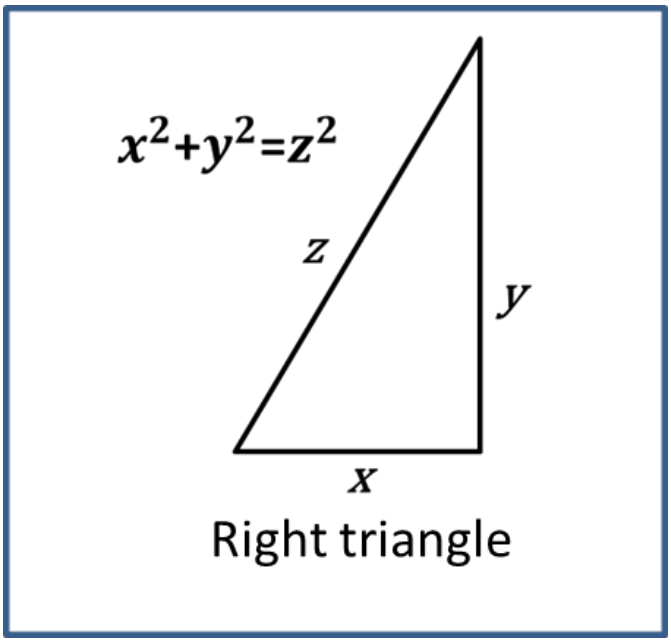

Figure 1. The right triangle satisfies: $z^2=x^2+y^2$

There are different methods to obtain the primitive Pythagorean triples; the best known is using equations (1) and (2). For the case that v is even and u is odd being $(u > v)$, is used equation (1) and equation (2) is used when (u, v) are odd being $(u > v)$.

$$\begin{aligned} z^2 &= x^2 + y^2 \\ x &= 2uv \\ y &= u^2 - v^2 \\ z &= u^2 + v^2 \end{aligned} \qquad (1)$$

$$\begin{aligned} z^2 &= x^2 + y^2 \\ x &= uv \\ y &= 1/2\,(u^2 - v^2) \\ z &= 1/2\,(u^2 + v^2) \end{aligned} \qquad (2)$$

Equations (1) and (2) were found in the following way:

If we look for the integer solutions of the equation $z^2 = x^2 + y^2$, it is the same as looking for the rational solutions of $\xi^n + \beta^n = 1$ for the special case of $n = 2$. (Figure 2), where $\xi = \frac{x}{z}$ and $\beta = \frac{y}{z}$ (dividing the equation $\xi^2 + \beta^2 = x^2 + y^2 = z^2$ by z^2).

Knowing a rational solution the $(x = 1, y = 0)$, from there we can deduce others, in this case all the solutions. The equation of the line that passes through (1,0) is:

$$\beta = k(\xi - 1)$$

Where k is the slope.

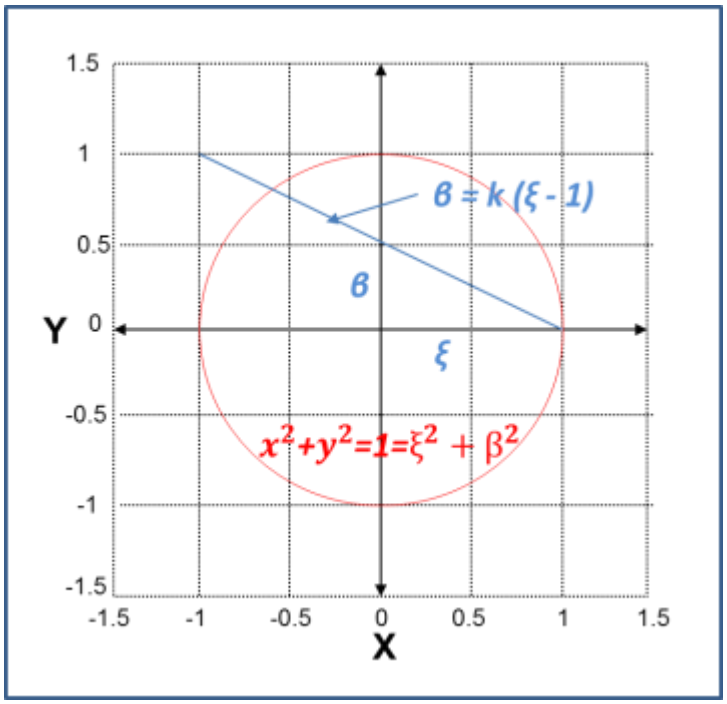

Figure 2. Intercept of the line $\beta = k(\xi - 1)$ with the circle $\xi^2 + \beta^2 = 1$..

Calculating the intersection of the line with the circle $\xi^2 + \beta^2 = 1 = x^2 + y^2$, (Figure 2) and replacing β of the equation of the line in the equation of the circle we have:

$$\xi^2 + k^2(\xi - 1)^2 = 1$$

$$\xi^2(1 + k^2) - 2k^2\xi + k^2 - 1 = 0$$

This second-degree equation has the solutions:

$$\xi = \frac{k^2}{1 + k^2} \pm \frac{1}{1 + k^2} = \frac{k^2 - 1}{1 + k^2} \qquad \text{———} \quad \text{———} \quad \text{———}$$

$$\beta = k(\xi - 1) = -\frac{2k}{1 + k^2} \qquad \text{———}$$

It is easy to see that in $\xi^2 + \beta^2 = 1$, β has two signs $(\pm)$. Since k is a negative rational number so that β is positive and quotient of two integers:

$k = -\frac{u}{v}$ – and the solution is:

$$\xi = \frac{u^2 - v^2}{u^2 + v^2} \qquad \text{———}$$

$$\beta = \frac{2uv}{u^2 + v^2} \qquad \text{———}$$

Recalling that $\xi = \frac{x}{z}$ – and $\beta = \frac{y}{z}$ – it would remain $(x = 2uv, y = u^2 - v^2, z = u^2 + v^2)$, which corresponds to equation (1) to find the primitive Pythagorean triples. For the case that (u, v) are primes and one of them is odd (u), the primitive Pythagorean triples obtained are known as the succession of prime numbers (2,3,5,7,11,13,17,19,23,29,31,37, ...) [4].

For the cases that are considered (u, v) odd primes the set of equations would be

$(x = uv, y = \frac{1}{2}(u^2 - v^2), z = \frac{1}{2}(u^2 + v^2))$, which corresponds to equation (2).

The biggest problem with the application of equations (1) and (2) when (u, v) are primes, is that it is required to know them, which is not so simple, although in [5] the pattern of the prime numbers was found, (previously considered to be random), its application requires the solution of several equations, so the method developed in section 3 is much simpler.

2. Definitions

2.1 Definition 1.

Primitive Pythagorean Triples: A Pythagorean triple consists of three positive integers (x, y, z) (x, y, z) that can be the two legs and the hypotenuse of a right triangle, thereby satisfying the equation $z^2 = x^2 + y^2$. This triple is considered primitive if (x, y, z) (x, y, z) do not share any common divisor other than 1. Each primitive triple is unique and does not repeat. A right triangle with sides that form a Pythagorean triple is known as a Pythagorean triangle.

2.2 Definition 2.

Tree of primitive Pythagorean triples: In mathematics, a tree of primitive Pythagorean triples is a data tree in which each node branches bifurcates to others three subsequent nodes with the infinite set of all nodes giving all (and only) primitive Pythagorean triples without duplication. This was first discovered by B. Berggren in 1934, [7] Figure 3.

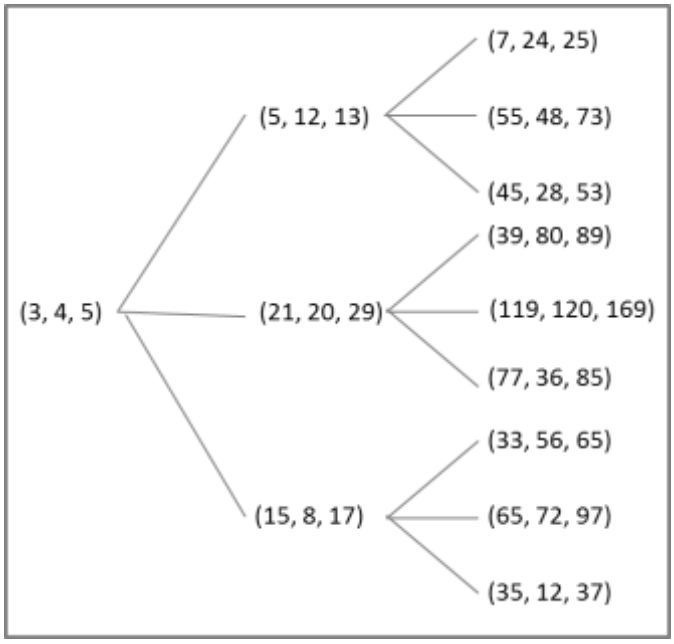

Figure 3. Pythagorean tree

2.3 Definition 3.

Coprimes: Two positive integers a and b are defined to be **coprime** (or **relatively prime**) if the greatest common divisor (gcd) of a and b is 1. Theoretically, this means that there is no integer greater than 1 that divides both a and b without leaving a remainder. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ two integers, some of them can differ from zero. The greatest common divisor (gcd) on $\boldsymbol{a}$ and $\boldsymbol{b}$ is the largest positive integer $\boldsymbol{d}$ noticed by $(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{d}$ that divides both $\boldsymbol{a}$ and $\boldsymbol{b}$. In the case in which $(\boldsymbol{a}, \boldsymbol{b}) = \boldsymbol{1}$, we say that $\boldsymbol{a}$ and $\boldsymbol{b}$ are coprimes.
When $(\boldsymbol{a}, \boldsymbol{b})$ are one even and the other odd and $\boldsymbol{a} = \boldsymbol{b} + \boldsymbol{1}$ or $(\boldsymbol{a}, \boldsymbol{b})$ are odd and $\boldsymbol{a} = \boldsymbol{b} + \boldsymbol{2}$, then ***(a, b) = 1***

Another way to generate the Primitive Pythagorean Triples

The Pythagorean Theorem says that in the following equation:

$$x^2 + y^2 = z^2 \qquad (3)$$

We can find integers $(x, y, z) > 0$, which meet the equation.

In equation (3), one of the two variables (x, y), must be greater than the other and can not be equal, because when they are equal I would give $z^2 = 2x^2 = 2y^2$and z would not be a integer, (the square root of 2 is an irrational number with an infinite mantissa, which when multiplied by a integer number, remains irrational).

Assuming that x is the smallest of the variables (x, y), then z must be greater than y, $(\boldsymbol{z > y})$**,** so that equation (3) has a solution, ie $z > y > x$, therefore, we can write that:

$$z = y + m \qquad \Rightarrow m \in Z^+ \qquad (4)$$

Where $x > m > 0$. If m equal to or greater than x, equation (3) would have no solution with equation (4).

Proof:

1. $z > y > x$, therefore $z = y + m$ y $z^2 = x^2 + y^2 = (y + m)^2$
2. Assume $m \geq x \Rightarrow m = x + u,\ u \geq 0$
3. $x^2 = 2ym + m^2$ ..... according to 1.
4. $x^2 = 2y(x + u) + (x + u)^2$, replacing m by $x + u$ ..... according to 2. and 3.
5. $x^2 = 2yx + 2yu + x^2 + 2xu + u^2$ ..... according to 4.
6. $0 = 2yx + 2yu + 2xu + u^2$ ..... according to 5.
7. $2yx = -2yu - 2xu - u^2$, it is an absurdity (a positive quantity cannot equate to a negative or to 0 when $u \geq 0$
8. Therefore, assumption 2. is incorrect ..... according to 7.
9. Therefore: $m < x$ ..... according to 8.

Being shown that $m < x$.

Figure 4 shows a graphical representation of equations (3) and (4). The importance of the equation (4) is that when $m \in Z^+$ being $m > 0$, we can find the Pythagorean triples,

because z would be between the natural numbers, the rest of values of $m \neq Z^+$, would give also $z \neq Z^+$.

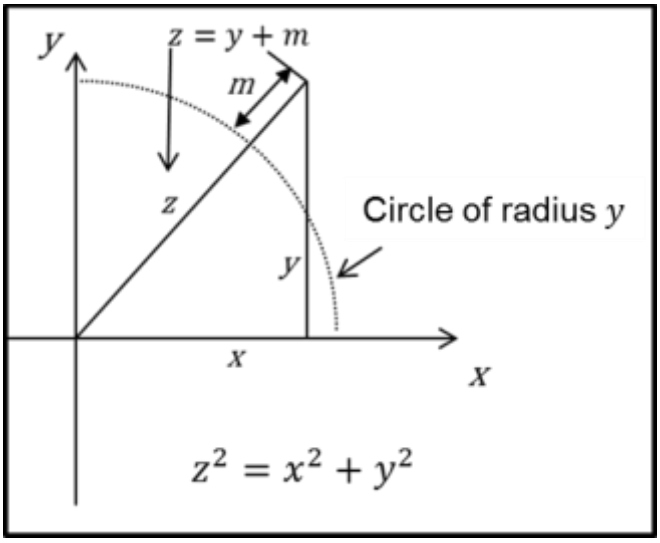

Figure 4. Graphic representation of equations (3) and (4).

Replacing equation (4) in equation (3) would be:

$$x^2 + y^2 = (y + m)^2 = y^2 + 2ym + m^2$$

$$\frac{x^2 - m^2}{2m} = y \qquad (5)$$

The primitive Pythagorean triples (x, y, z) with $x \geq 3$ odd, are replacing $m = 1$ in equations (4) and (5), being:

$$z = y + 1 \qquad (6)$$

$$\frac{x^2 - 1}{2} = y \qquad (7)$$

The primitive Pythagorean triples (x, y, z) with $x \geq 4n$, $n \geq 1$ even, are replacing $m = 2$ in equations (4) and (5), being:

$$z = y + 2 \qquad (8)$$

$$\frac{x^2 - 4}{4} = (2n)^2 - 1 = y \qquad (9)$$

Equations (6), (7), (8) and (9) not only allow to find primitive Pythagorean triples (x, y, z) but also is checked with equations (6) and (7), that any odd number x equal to or greater than $3 + 2n \Rightarrow n \geq 0$, belongs to one of the values of a primitive Pythagorean triple already found by the Pythagoreans.

Proof:

1. $x = 3 + 2n \Rightarrow n \geq 0$ (x is always odd)
2. $y = (x^2 - 1)/2$, equation (7)

3. Replacing 1. in 2. we have:

$y = ((3 + 2n)^2 - 1)/2$

$y = (9 + 12n + 4n^2 - 1)/2$

$y = 4 + 6n + 2n^2$=$2(2 + 3n + n^2)$ which is always even for any $n \geq 0$.

4. $z = y + 1$, equation (6), therefore z is odd and (y, z) are coprimes to be separated from each other by 1 only.

5. According to 4. (x, y) are also coprimes (x odd, y even), if there were a common divisor greater than 1, this would also be divisor of z, which contradicts what is found in 4. therefore the Pythagorean triples with $x = 3 + 2n \Rightarrow n \geq 0$, $y = (x^2 - 1)/2$, $z = y + 1$ are primitives.

The proof is shown.

The primitive Pythagorean triples (x, y, z) give origin to integer Pythagorean triples but greater than the previous ones according to the equation (10), where $a = xk, b = yk, c = zk$ being $k \geq 1$, that is to say (a, b, c) have a common divisor k, when $k = 1$ would be primitive Pythagorean triples and when $k > 1$ would be major Pythagorean triples (not primitive):

$$(xk)^2 + (yk)^2 = (zk)^2 \quad (10)$$

When the primitive Pythagorean triples are obtained with the equations (6) and (7) and the equation (10) is applied to obtain the major Pythagorean triples $m = k$ and when the primitive Pythagorean triples are obtained with the equations (8) and (9) and equation (10) is applied to obtain the major Pythagorean triples $m = 2k$.

Tables 1 and 2 show examples of the above.

Table 1. Primitive and mayor Pythagorean triples with x odd and $m = k$ and $1 \leq k \leq 4$

Primitive Pythagorean triples x odd $m = k$ and $k = 1$, equations (6), (7) and (10)									
x	3	5	7	9	11	13	15	17	19
y	4	12	24	40	60	84	112	144	180
z	5	13	25	41	61	85	113	145	181

Mayor Pythagorean triples x odd $m = k$ and $k = 2$, equations (6), (7) and (10)									
$a = xk$	6	10	14	18	22	26	30	34	38
$b = yk$	8	24	28	80	120	168	224	288	360
$c = zk$	10	26	50	82	122	170	226	290	362

Mayor Pythagorean triples x odd $m = k$ and $k = 3$, equations (6), (7) and (10)									
$a = xk$	9	15	21	27	33	39	45	51	57
$b = yk$	12	36	72	120	180	252	336	432	540
$c = zk$	15	39	75	126	183	255	339	435	543

Mayor Pythagorean triples x odd $m = k$ and $k = 4$, equations (6), (7) and (10)									
$a = xk$	12	20	28	36	44	52	60	68	86
$b = yk$	16	48	96	160	240	336	448	576	720
$c = zk$	20	52	100	164	244	340	452	580	724

Table 2. Primitive and mayor Pythagorean triples with x even, $m = 2k$ and $1 \leq k \leq 4$

Primitive Pythagorean triples x even $m = 2k$ and $k = 1$, equations (8), (9) and (10)									
x	8	12	16	20	24	28	32	36	40
y	15	35	63	99	143	195	255	323	399
z	17	37	65	101	145	197	257	325	401

Mayor Pythagorean triples x even $m = 2k$ and $k = 2$, equations (8), (9) and (10)									
$a = xk$	16	24	32	40	48	56	64	72	80
$b = yk$	30	70	126	198	286	390	510	646	798
$c = zk$	34	74	130	202	190	394	514	650	802

Mayor Pythagorean triples x even $m = 2k$ and $k = 3$, equations (8), (9) and (10)									
$a = xk$	24	36	48	60	72	84	96	108	120
$b = yk$	45	105	189	297	429	585	765	969	1197
$c = zk$	51	111	195	303	435	601	771	975	403

Mayor Pythagorean triples x even $m = 2k$ and $k = 4$, equations (8), (9) and (10)									
$a = xk$	32	48	64	80	96	112	128	144	160
$b = yk$	60	140	252	396	572	780	1020	1292	1596
$c = zk$	68	148	260	404	580	788	1028	1300	1604

The primitive Pythagorean triples with x even and $m = 2$, can only be obtained for $x = 4n$ being $n \geq 1$, this because the even numbers $x = 2 + 4n$ being $n \geq 1$, are derived from the primitive Pythagorean triples with x odd and $k > 1$,, which generates major Pythagorean triples (see table 1) and that with $x = 4n$ always x is even being y odd and therefore (x, y) are coprimos generating primitive Pythagorean triples.

Proof:

1. $x = 4n \Rightarrow n \geq 0$ (always x is even)

2. $y = \frac{x^2 - m^2}{2m}$

3. Replacing $m = 2$ and $x = 4 + 4n$ en 2. then:

4. $y = \frac{(4n)^2 - 4}{4} = 4(1 + n)^2 - 1$

5. According to 4. y is always odd.

6. $z = y + 2$, therefore z is odd and (y, z) are coprimes, (two odd numbers separated from each other by 2 are always coprimes.

7. According to 6. (x, y) are also coprimes (x even, y odd), if there were a common divisor greater than 1, this would also be a divisor of z, which contradicts what is found in 6. therefore the Pythagorean triples with $x = 4n \Rightarrow n \geq 0$, $y = (x^2 - 4)/4$, $z = y + 2$ are primitive.

The proof is shown.

In equations (1) and (2) the value of m can be obtained as follows:

$z = u^2 + v^2 = y + m$

$y = u^2 - v^2$

$z = u^2 + v^2 = u^2 - v^2 + m$

$m = z - y = 2v^2$

When v is a eve prime, being that the only even prime is 2, the value of m will be $m = 2v^2 = 8$ and equation (1) would be in equation (11):

$z^2 = x^2 + y^2$

$$x = 4u$$

$y = u^2 - 4$ (11)

$m = 8$

$z = u^2 + 4 = y + 8$

Table 3 shows examples of primitive Pythagorean triples with equation (11) where $m = 8$ and u is an odd prime.

Table 3. Primitive Pythagorean Triples with Equation (11), where $m = 8$ and u is an odd prime

Primitive Pythagorean Triples with equation (11), $m = 8, x = 4u, y = u^2 - 4, z = y + 8$									
u	3	5	7	11	13	17	19	23	29
x	12	20	28	44	52	68	76	92	116
y	5	21	45	117	165	285	357	525	837
z	13	29	53	125	173	293	365	533	845

Here, for example, it is interesting to analyze the form that z could have within the set of natural numbers. In Porras-Ferreira [8], it is shown that all primes have the following form:
$p_m = [1,7,11,13,17,19,23,29] + 30n$, for $n \geq 0$, except for 1 that is not prime for $n = 0$ and 2, 3 and 5 that are formed independently. Since z is equal to the sum of two prime numbers squared, where one of them is even, with 2 being the only even prime, all z endings will be in 3 or 5, therefore their solutions for $z \geq 30$ will have the form of equation (12):

$$z \equiv [\,^{5\ mod\ 30}\,] = \{\,^{23\ mod\ 30} \quad \begin{matrix} 5 + 30n\ \{\text{Composites}\}\ \text{fore some}\ n \geq 4 \\ 23 + 30n\ \{\text{Primes or composites}\}\ \text{for some}\ n \geq 1 \end{matrix} \quad (12)$$

Likewise, for example with $a = 24$, there are only four solutions of $b > a$ between the natural numbers so that c is a natural number and they are $b = [143, 70,45,32]$ and $c = [145, 74, 51, 40]$ respectively, all values obtained with the equations (6), (7), (8), (9) and (10), solutions with $m = k$ and $k = 8$ and $m = 2k$ and $k = [1, 2, 3]$ and with equation (11) a solution with $m = 8k$ and $k = 2$, $b = 10$ and $c = 26$ where all k are divisors of 24. Note that this last solution has $b < a$, which tells us that if we set x to find all the possible solutions of integer numbers of equation (3) where $y < x > m$, we can vary m, independently as long as equations (4) and (5) are met, which makes the proposed method be even more interesting to find any Pythagorean triples.

For the cases where u and v are primes and both are odd, primitive Pythagorean triples can be obtained by equation (13):

$$z^2 = x^2 + y^2$$
$$x = uv$$
$$y = 1/2\ (u^2 - v^2)$$
$$z = 1/2\,(u^2 + v^2) \qquad (13)$$

$$z = y + m$$
$$m = v^2$$
$$u > v$$

Table 4 gives examples using equation (13)

Table 4. Primitive Pythagorean Triples with Equation (13), where $m = v^2$ and (u, v) are odd numbers or primes

Pythagorean primitive terns with equation (13), (u, v) primes									
u	5	7	11	13	17	19	23	29	31
v	3	3	3	3	3	3	3	3	3
m	9	9	9	9	9	9	9	9	9
x	15	21	33	39	51	57	69	87	93
y	8	20	56	80	140	176	260	416	476
$z = y + m$	17	29	65	89	149	185	269	425	485

All integers (x, y) that are not within the Pythagorean triples (primitive and / or, mayor), the calculation of z in equation (3) is irrational with infinite mantissa, its calculation comes from a root of 2, $z = \sqrt[2]{x^2 + y^2}$, that is, m in $z = y + m$ is irrational with an infinite mantissa.

Conclusions

In summary, our method offers a fresh perspective on generating primitive Pythagorean triples, complementing the ancient knowledge of the Pythagorean Theorem and Fibonacci's demonstration. Through equations (4) and (5), we've devised a straightforward approach to obtaining these triples, confirming the Pythagoreans' discovery regarding odd numbers and uncovering a new insight into the inclusion of even numbers of the form $x = 4 + 4n \Rightarrow$ where $n \geq 0 n \geq 0$. Moreover, our method facilitates the determination of the number of integer solutions for a given value of x, enhancing our understanding of the distribution of primitive Pythagorean triples. Additionally, we've established direct connections between equations (1) and (2) with the parameter mm, leading to the derivation of equations (11) and (13) for calculating primitive Pythagorean triples. This comprehensive approach not only simplifies the generation process but also unveils deeper connections within the realm of Pythagorean triples, advancing our knowledge in this fundamental area of mathematics.

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