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ANALYTICAL BEHAVIOUR OF NEW MODELLING OF OFFSET FRACTIONAL FOURIER TRANSFORM

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The Fractional Fourier Transform (FrFT) is a generalization of the classical Fourier transform that provides a continuous transition between the time and frequency domains through a fractional-order rotation in the time-frequency plane. Several researchers have investigated its analytical properties and extensions to multidimensional and generalized fractional transforms. Motivated by these developments, the present study investigates the analyticity properties of New Modelling of Offset fractional Fourier transform. The work aims to extend the understanding of the analytical behaviour and mathematical structure of fractional transform operators.

Keywords: Fractional Fourier transform, Offset fractional Fourier transform, testing function space.

I. INTRODUCTION:

The Fourier transform is one of the fundamental mathematical tools in harmonic analysis and signal processing. It provides a means of representing a function in the frequency domain and plays an essential role in the study of differential equations, integral equations, and various problems in applied mathematics [1], [2]. However, the classical Fourier transform provides only a fixed representation in the time frequency domain. To overcome this limitation, the fractional Fourier transform (FrFT) was introduced as a generalization of the classical Fourier transform, allowing a continuous transition between the time and frequency domains. The fractional Fourier transform has attracted considerable attention because of its mathematical structure and its applications in optics, quantum mechanics, signal processing, and communication theory.

The Fractional Fourier Transform (FrFT) can be understood geometrically as a generalized rotational operation in the time–frequency plane, in which a signal undergoes a rotation through an arbitrary angle. The fractional order determines the angle of rotation, and the classical Fourier transform is recovered as a special case. Mendlovic, Ozaktas, and Lohmann demonstrated the correspondence between light propagation in graded-index media and the rotational behavior of the Wigner distribution, thereby providing a deeper physical understanding of the fractional Fourier transform [3], [4]. This interpretation provides a useful framework for studying time frequency representations and establishes important connections between Fourier analysis and fractional integral operators. Contributions by V D Sharma and S A Khapre [5] concerned the analyticity of the

generalized two-dimensional fractional cosine transform. Their work extended the investigation of complex analytic properties to another class of fractional integral transformations and provided additional mathematical background for studying analyticity in multidimensional fractional transform domains. In related work, Sharma and Thakare [6] investigated operational calculus for the two-dimensional Offset Fractional Fourier Transform, deriving operator-related properties within the generalized distributional framework. Analyticity of distributional Fourier-Laplace transform has been checked in [7].

Further developments of the fractional Fourier transform have led to the introduction of the Offset fractional Fourier transform (FrFT), which incorporates additional parameters to provide greater flexibility in signal representation. The offset parameters introduce translations or modifications in the associated time-frequency representation, depending on the adopted definition of the transform. Consequently, the Offset FrFT provides a generalized framework that extends the conventional fractional Fourier transform and offers additional mathematical possibilities for studying signals and functions. Some recent combiations on fractional Fourier transform can be seen in work of H. Shi, H. Yang, and Y. Qiao in [8].

From the perspective of mathematical analysis, integral transforms are closely connected with the theory of analytic functions. Analyticity is a fundamental concept in complex analysis and plays an important role in understanding the behavior of complex-valued functions. A function is said to be analytic in a domain if it possesses a complex derivative at every point of that domain. The study of analyticity provides information about the regularity, convergence, differentiability, and growth properties of functions. In particular, the analyticity of an integral transform depends on the analytical structure of its kernel and the integrbility and growth conditions imposed on the original function.

II. THE TESTING FUNCTION SPACE E($\mathbf{R^n}$)

An infinitely differentiable complex valued function ϕ on R^n belongs to $E(R^n)$, if for each compact set $I \subset S_{a,b}$, where

$$S_{a,b} = \{x, y : x, y \in R^n, |x| \leq a, |y| \leq b, a > 0, b > 0\},$$

$$\gamma_{E_{m,n}}(\phi) = \sup_{x,y \in I} \left| D_{x,y}^{m,n} \phi(x,y) \right| < \infty \ , \ \ m, n = 0,1,2, \ldots \ldots$$

Thus, $E(R^n)$ will denote the space of all $\phi \in E(R^n)$ with compact support contained in $S_{a,b}$.

III. THEOREM

To prove for each $x, y \in R^n$ and $0 < \alpha \leq \frac{\pi}{2}$, the function $K_\alpha(x, u-\eta, y, p)$ is belongs to $E(R^n)$ as function of x and y, where

$$K_\alpha(x, u-\eta, y, p) = \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\, y^{p-1} e^{iu\tau} e^{\frac{i}{2\sin\alpha}\{[x^2 + (u-\eta)^2]\cos\alpha - 2[x(u-\eta)]\}}$$

Proof: Consider

$$\gamma_{E_{m,n}}[K_\alpha(x,u-\eta,y,p)] = \sup_{x,y\in I}\left|D_{x,y}^{m,n}K_\alpha(x,u-\eta,y,p)\right|$$

$$= \sup_{x,y\in I}\left|D_{x,y}^{m,n}\sqrt{\frac{1-icot\alpha}{2\pi}}y^{p-1}\mathrm{e}^{\mathrm{i}(u\tau)}e^{\frac{i}{2sin\alpha}\{[x^2+(u-\eta)^2\,]cos\alpha-2[x(u-\eta)]\}}\right|$$

$$= sup_{x,y\in I}\left|C_{1\alpha}\mathrm{e}^{\mathrm{i}u\tau}\sum_{k=0}^{m}C_m\;C_\alpha\big(x\cos\alpha-2(u-\eta)\big)^{m-2k}e^{(m-k)q}(p-n)\,!\,y^{p-n-1}\right|$$

$$< \infty,$$

for any fixed $x, y \in R^n$ and any fixed m, n, $0 < \alpha \leq \frac{\pi}{2}$.

Thus, function $K_\alpha(x,u-\eta,y,p) \in E(R^n)$.

IV. GENERALIZED NEW MODELLING OF OFFSET FRACTIONAL FOURIER TRANSFORM.

The distributional New Modelling of Offset fractional Fourier transform of $f(x,y) \in E^*(R^n)$ is defined by,

$$F_\alpha^{\tau,\eta}\{f(x,y)\}(u,p) = F_\alpha(u,p) = \langle f(x,y),\ K_\alpha(x,u-\eta,y,p))\rangle, \quad \ldots\ldots\ldots\ldots \quad (1)$$

where

$$K_\alpha(x,u-\eta,y,p) = \sqrt{\frac{1-icot\alpha}{2\pi}}y^{p-1}e^{iu\tau}e^{\frac{i}{2sin\alpha}\{[x^2+(u-\eta)^2\,]cos\alpha-2[x(u-\eta)]\}}$$

..... (2)

$$= C_{1\alpha}y^{p-1}\mathrm{e}^{\mathrm{i}(u\tau)}e^{iC_{2\alpha}\{[x^2+(u-\eta)^2\,]cos\alpha-2[x(u-\eta)]\}},$$

where $C_{1\alpha} = \sqrt{\frac{1-icot\alpha}{2\pi}}$, $C_{2\alpha} = \frac{1}{2sin\alpha}$

Here right hand side of equation (1) is meaningful because according to above theorem equation (1), $K_\alpha(x, u-\eta, y, p) \in E$ and $f \in E^*$.

Moreover, we say that f is a Offset fractional Fourier-Mellin transformable, if it is a member of E^*, the dual space of E.

V. Analyticity theorem

Let $f(x,y) \in E^*(R^n)$ and its generalized new modelling of Offset fractional Fourier transform be defined by,

$M\ F_\alpha^{\tau,\eta}\{f(x,y)\}(u,p) = F_\alpha(u,p) = \langle f(x,y),\ K_\alpha(x,u-\eta,y,p))\rangle$ then $F_\alpha(u,p)$ is analytic on C^n

if $suppf \subset S_{a,b}$,

where $S_{a,b} = \{x,y: x,y \epsilon R^n, |x| \le a, |y| \le b, a>0, b>0\}$

Moreover, $F_\alpha(u,p)$is differentiable and

$$D_{x,y}^{m,n}\{F_\alpha(u,p)\} = \langle f(x,y),\quad D_{x,y}^{m,n}K_\alpha(x,u-\eta,y,p)\}\rangle.$$

Proof: Let $u: (u_1, u_2, ---, u_j, ---u_n) \epsilon\ C^n$ and $p: (p_1, p_2, ---, p_j, ---p_n) \epsilon\ C^n$

We first prove that

$$\frac{\partial}{\partial u_j}\{F_\alpha(u,p)\} = \langle f(x,y), \frac{\partial}{\partial u_j}\ K_\alpha(x,u-\eta,y,p)\rangle,$$

For fixed $u_j \neq 0$, choose two concentric circles C and C' with centre at $u_j$with radii r and r_1 respectively such that $0 < r < r_1 < |u_j|$.

Let Δu_j be a complex increment satisfying $0 < |u_j| < r$.

Consider,

$$\frac{F_\alpha(u_j+\Delta u_j, p) - F_\alpha(u_j, p)}{\Delta u_j} - \langle f(x,y), \frac{\partial}{\partial u_j}K_\alpha(x,u-\eta,y,p)\rangle$$

$$= \langle f(x,y),\ \psi_{\Delta u_j}(x,y)\rangle \qquad \text{..... (3)}$$

$$\Rightarrow \frac{1}{\Delta u_j}\{\langle f(x,y), K_\alpha(x,(u_j+\Delta u_j)-\eta,y,p)\rangle - \langle f(x,y), K_\alpha(x,u_j-\eta,y,p)\rangle\}$$

$$-\langle f(x,y), \frac{\partial}{\partial u_j}K_\alpha(x,u-\eta,y,p)\rangle = \langle f(x,y), \psi_{\Delta u_j}(x,y)\rangle$$

$$\Rightarrow <f(x,y),\ \frac{1}{\Delta u_j}\{\ K_\alpha(x,\ (u_j+\Delta u_j)-\eta,y,p) - K_\alpha(x,u_j-\eta,y,p)\}$$

$$-\frac{\partial}{\partial u_j}K_\alpha(x,u-\eta,y,p) >= \langle f(x,y),\ \psi_{\Delta u_j}(x,y)\rangle,$$

where

$$\psi_{\Delta u_j}(x,y) = \frac{1}{\Delta u_j}\{\ K_\alpha(x,(u_j+\Delta u_j)-\eta,y,p) - \ K_\alpha(x,u_j-\eta,y,p)\}$$

$$-\frac{\partial}{\partial u_j}K_\alpha(x,u-\eta,y,p).$$

For any fixed $x, y \epsilon R^n$ and fixed integer $m = (m_1, m_2, - - - - m_n)$,

$D_x^m K_\alpha(x,u-\eta,y,p)$

$$= D_x^m\left\{\sqrt{\frac{1-icot\alpha}{2\pi}}\,y^{p-1}e^{iu\tau}e^{\frac{i}{2sin\alpha}\{[x^2+(u-\eta)^2]cos\alpha-2[x(u-\eta)]\}}\right\}$$

$$= C_{1\alpha}e^{i(u\tau)}\left\{D_x^m e^{iC_{2\alpha}\{[x^2+(u-\eta)^2]cos\alpha-2x(u-\eta)\}}\right\}$$

$$= C_{1\alpha}e^{i(u\tau)}\beta\ D_x^m\{\ e^{iC_{2\alpha}\{[x^2+(u-\eta)^2]cos\alpha-2x(u-\eta)\}}\},\quad \text{where } \beta = y^{p-1}$$

$$= C_{1\alpha}e^{i(u\tau)}\beta\ \sum_{k=0}^{m} C_m C_\alpha\left(xcos\alpha-(u-\eta)\right)^{m-2k}e^{(m-k)q},$$

where $C_m = \frac{m!}{k!\,(m-2k)!}2^{m-2k}(i)^{m-k}$, $C_\alpha = (C_{2\alpha})^{m-k}(cos\alpha)^k$

and $q = iC_{2\alpha}[(x^2+(u-\eta)^2)cos\alpha - 2x(u-\eta)]$

Since for any fixed $x \in R^n$, fixed integer m and $0 < \alpha \leq \frac{\pi}{2}$,

$D_x^m K_\alpha(x,u-\eta,y,p)$ is analytic inside and on C',

we have by Cauchy's integral formula,

$$D_x^m\psi_{\Delta u_j}(x,y) = \frac{1}{2\pi i}D_x^m\int_{C'}K_\alpha(x,\widetilde{u-\eta},y,p)\left[\frac{1}{\Delta u_j}\left(\frac{1}{z-u_j-\Delta u_j}-\frac{1}{z-u_j}\right)\right.$$

$$-\frac{1}{(z-u_j)^2}\Bigg]dz,$$

where $\tilde{u} = u_1, u_2, - - -, u_{j-1}, z, u_{j+1}, - - - u_n$

$$= \frac{1}{2\pi i} D_x^m \int_{C'} K_\alpha(x, \widetilde{u-\eta}, y, p) \left[\frac{1}{\Delta u_j}\left(\frac{z-u_j-z+u_j+\Delta u_j}{(z-u_j-\Delta u_j)(z-u_j)}\right) - \frac{1}{(z-u_j)^2}\right] dz$$

$$= \frac{1}{2\pi i} D_x^m \int_{C'} K_\alpha(x, \widetilde{u-\eta}, y, p) \left[\left(\frac{1}{(z-u_j-\Delta u_j)(z-u_j)}\right) - \frac{1}{(z-u_j)^2}\right] dz$$

$$= \frac{1}{2\pi i} D_x^m \int_{C'} K_\alpha(x, \widetilde{u-\eta}, y, p) \left[\frac{z-u_j-z+u_j+\Delta u_j}{(z-u_j-\Delta u_j)(z-u_j)^2}\right] dz$$

$$= \frac{\Delta u_j}{2\pi i} \int_{C'} D_x^m \left[\frac{K_\alpha(x, \widetilde{u-\eta}, y, p)}{(z-u_j-\Delta u_j)(z-u_j)^2}\right] dz$$

$$= \frac{\Delta u_j}{2\pi i} \int_{C'} \left[\frac{P(x, \widetilde{u-\eta}, y, p)}{(z-u_j-\Delta u_j)(z-u_j)^2}\right] dz$$

But for all $z \epsilon C'$ and x is restricted to a compact subset of R^n, $0 < \alpha \leq \frac{\pi}{2}$,

$P(x, \widetilde{u-\eta}, y, p) = D_x^m K_\alpha(x, \widetilde{u-\eta}, y, p)$is bounded by a constant L.

Moreover, $|z-u_j-\Delta u_j| > r_1 - r > 0$ & $|z-u_j| = r_1$.

Therefore we have,

$$\left|D_x^m \psi_{\Delta u_j}(x,y)\right| = \left|\frac{\Delta u_j}{2\pi i} \int_{C'} \frac{P(x, \widetilde{u-\eta}, y, p)}{(z-u_j-\Delta u_j)(z-u_j)^2} dz\right|$$

$$\leq \frac{|\Delta u_j|}{2\pi} \int_{C'} \frac{L}{(r_1-r){r_1}^2} |dz|$$

$$\leq \frac{|\Delta u_j|}{2\pi} \frac{L}{(r_1-r){r_1}^2} \int_{C'} |dz|$$

$$\leq \frac{|\Delta u_j|}{2\pi} \frac{L}{(r_1-r){r_1}^2} 2\pi r_1$$

$$\leq \frac{|\Delta u_j| L}{(r_1 - r) r_1}$$

Similarly $\left|D_y^n \psi_{\Delta v_j}(x,y)\right| \leq \frac{|\Delta v_j| N}{(r_1 - r) r_1}$,

where $Q(x, u-\eta, y, \tilde{p}) = D_y^n K_\alpha(x, u-\eta, y, \tilde{p})$is bounded by a constant N.

Thus as $|\Delta u_j| \to 0$ and $|\Delta p_j| \to 0$, $D_x^m \psi_{\Delta u}(x,y)$ and $D_y^n \psi_{\Delta p_j}(x,y)$tends to zero uniformly on the compact subset of R^n, therefore it follows that $\psi_{\Delta u_j}(x,y)$ and $\psi_{\Delta p_j}(x,y)$ converges in $E(R^n)$ to zero. Since $f(x,y) \in E^*$, we conclude that equation (3) also tends to zero.

Therefore, $F_\alpha(u,p)$is differentiable with respect to u_j and v_j. But this is true for all $j = 1,2, \ldots n$.

Hence $F_\alpha(u,p)$ is analytic on C^n and

$D_{x,y}^{m,n} F_\alpha\{f(x,y)\}(u,p) = \langle f(x,y),\ D_{x,y}^{m,n} K_\alpha(x, u-\eta, y, p\}\rangle$.

VI. Conclusion:

Fractional Fourier transform provides a broader mathematical framework for understanding the relationship between classical Fourier analysis and generalized fractional integral transforms. In this paper analyticity theorem of New Modelling of Offset fractional Fourier transform is proved. Its analytical properties, particularly analyticity, play an important role in extending transform theory to multidimensional and distributional settings. Thus, the study of analyticity in fractional transform domains offers scope for further investigations in harmonic analysis, generalized function theory, and their applications in applied mathematics.

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